1.1 Functions of Bounded Variation

Def. Let $f:[a,b]\to\mathbb{R}$, $P=(x_0,x_1,\dots x_m)$ a partition of $[a,b]$. Define

$$V_P=\sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|$$

Define $V=V_{[a,b]}=\sup_P\{V_P \mid P \text{ a partition of } [a,b]\}$

Clearly $0\le V\le\infty$. If $V\neq\infty$, ($V=\infty$) we say f is of bounded variation (unbounded variation) on $[a,b]$.

Ex. If $f(x)=$ constant, then $V_{[a,b]}=0$

Ex. If $f(x)$ is monotone, $V_{[a,b]}=|f(b)-f(a)|$

Suppose f is decreasing. Then

$$V_P=\sum_{i=0}^{n}|f(x_i)-f(x_{i-1})|=\sum_{i=1}^{m}f(x_{i-1})-f(x_i)$$
$$=\underbrace{\sum_{i=1}^{m}f(x_{i-1})}_{x_0 \text{ to } x_{m-1}}-\underbrace{\sum_{i=1}^{m}f(x_i)}_{x_1 \text{ to } x_m}=f(x_0)-f(x_m)=|f(b)-f(a)|$$

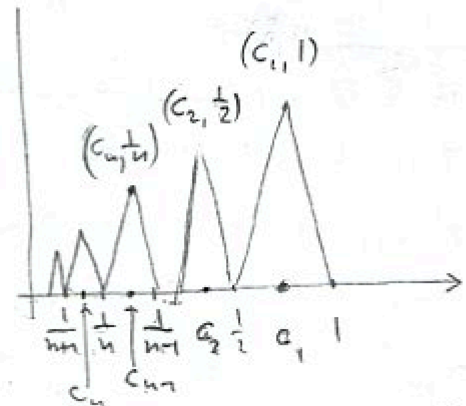
Ex. $f(x)=\begin{cases}1, & \text{if } x=0\\ 0, & \text{if } x\neq 0\end{cases}$

If $0\in[a,b]$, then $V_P=0$ (no part. pts $=0$)
$=2$ (a part pt $=0$)

so $V_{[a,b]}=2$

If $0\notin[a,b]$, $V_{[a,b]}=0$

Ex:



$P=(0,c_n,\frac1n,c_{n-1},\frac{1}{n-1},\dots,c_1,1)$

$V_P=\frac1n+\frac1n+\frac{1}{n-1}+\frac{1}{n-1}+\dots+\frac12+\frac12+1+1$
$=2\sum_{i=1}^{m}\frac1i \leftarrow$ unbounded

n can be chosen to be any, and $\{\sum_{i=1}^{m}\frac1i, i\in\mathbb{N}\}$ is unbounded, so $V_{[0,1]}=\infty$ (and it's a continuous function on $[0,1]$!)

Ex. $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$

is of unbounded variation.

Ex. If f is Lipschitz ($\exists C\geq 0$ $|f(x)-f(y)|\le C|x-y|$) then it is of bounded variation.

In particular, if f has a continuous first derivative, then it is Lipschitz.

Theorem 1.1

(i) If f is of bounded variation on $[a,b]$, then f is bounded on $[a,b]$.

(ii) If f, g are of bounded variation on $[a,b]$, then so are cf, $f \pm g$, $f \cdot g$. Furthermore, if there is a number $\varepsilon > 0$ s.t. $|g(x)| \geq \varepsilon$ on $[a,b]$, then $\frac{f}{g}$ is of bounded variation.

Theorem 1.2 Let $f: [a,b] \to \mathbb{R}$

(i) If $[a',b'] \subseteq [a,b]$, then $V_{[a',b']} \leq V_{[a,b]}$

(ii) If $a < c < b$, then $V_{[a,b]} = V_{[a,c]} + V_{[c,b]}$

Pf: (i) Let P' be a partition of $[a',b']$, and let $P = P' \cup \{a,b\}$ (add a, b to partition, if they are not already there). Then

$$V_{[a,b]} \geq V_P = |f(x_0) - f(a)| + \underbrace{\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|}_{V_{P'}} + |f(b) - f(x_m)|$$

So for any partition P' of $[a',b']$, $V_{[a,b]}$ is an upper bound. Then for $V_{[a',b']} = \sup\{V_{P'} \mid P'\}$ we have $V_{[a',b']} \leq V_{[a,b]}$

(ii) Let P_1, P_2 be partitions of $[a,c]$, $[c,b]$.

$P = P_1 \cup P_2$

Then $V_P = V_{P_1} + V_{P_2} \Rightarrow V_{P_1} + V_{P_2} \leq V_{[a,b]}$

for every partition P_1 of $[a,c]$, P_2 of $[c,b]$

Thus, $V_{P_1} \leq V_{[a,b]} - V_{P_2}$ for any P_1

$\Rightarrow V_{[a,c]} \leq V_{[a,b]} - V_{P_2}$ for any P_2

$V_{P_2} \leq V_{[a,b]} - V_{[a,c]}$ for any P_2

$\Rightarrow V_{[c,b]} \leq V_{[a,b]} - V_{[a,c]}$, so

$V_{[a,c]} + V_{[c,b]} \leq V_{[a,b]}$.

Let P be a partition of $[a,b]$, $P' = P \cup \{c\}$, $P_1 = P' \cap [a,c]$, $P_2 = P' \cap [c,b]$

Then $V_P \leq V_{P'} = V_{P_1} + V_{P_2} \leq V_{[a,c]} + V_{[c,b]}$

Since this is true for any P, we get

$V_{[a,b]} \leq V_{[a,c]} + V_{[c,b]}$.

Def. For any $x \in \mathbb{R}$, let

$$x^+ = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \qquad x^- = \begin{cases} 0, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Clearly $x^+, x^- \geq 0$, $|x| = x^+ + x^-$, $x = x^+ - x^-$

$x^+ = \frac{1}{2}(|x| + x)$, $x^- = \frac{1}{2}(|x| - x)$

Given a partition P of $[a,b]$ define

$$P_P = \sum_{i=1}^{m} \left(f(x_i) - f(x_{i-1})\right)^+$$

$$N_P = \sum_{i=1}^{m} \left(f(x_i) - f(x_{i-1})\right)^-$$

Then $V_P = P_P + N_P$, $P_P - N_P = f(b) - f(a)$

Define $P = \sup\{P_P \mid P \text{ a part. of } [a,b]\}$

$N = \sup\{N_P \mid P \text{ a part. of } [a,b]\}$

Theorem 1.6 If one of P, N, V is finite, so are the others. In this case,

$$V = P + N, \quad P - N = f(b) - f(a)$$

or $P = \frac{1}{2}(V + f(b) - f(a))$, $N = \frac{1}{2}(V - f(b) + f(a))$

Pf. Note that P finite $\Leftrightarrow$ N is finite due to

$P_\rho - N_\rho = f(b) - f(a)$ (differ by a const)

$\Rightarrow P - N = f(b) - f(a)$ $(\sup(a+S) = a + \sup S)$

Now, $P_\rho + N_\rho = V_\rho \le V \Rightarrow$ If V is finite, so are P, N

Also, $V_\rho = P_\rho + N_\rho \le N + P \Rightarrow$ If one of P, N is finite, so is the other, so $V \le P + N$

Now $P_\rho + N_\rho \le V$ for any ρ

$2P_\rho - (f(b) - f(a)) \le V$ | take supremum over all ρ

$2P - (f(b) - f(a)) \le V$

so $P + N \le V$

We get $P + N = V$.

Corollary 1.7 (Jordan's Theorem) f is of bounded variation on $[a,b]$ iff it is a difference of two bounded, increasing functions on $[a,b]$.

Pf. $\Leftarrow$ obvious by 1.1

$\Rightarrow$) If f is of bounded variation, then it is of bounded variation on any $[a,x]$ $x \in [a,b]$

exist by 1.6

Set $P(x) = P_{[a,x]}$, $N(x) = N_{[a,x]}$

$P(x)$, $N(x)$ are increasing by the facts:

$P_{[a',b']} \le P_{[a,b]}$, $N_{[a',b']} \le N_{[a,b]}$ when $[a',b'] \subseteq [a,b]$

Now: $P(x) - N(x) = f(x) - f(a)$, so

$$f(x) = \underbrace{P(x) + f(a)}_{\text{incr, bounded}} - N(x)$$

Theorem 1.8 Every function of bounded variation has at most countably many discontinuities.

Pf. By 1.7, $f = f_1 - f_2$, f_1, f_2 bounded and increasing. By 5.6.4, each f_i has countably many discontinuities, at set $A_i \subseteq [a,b]$. Then the set of discontinuities of f is contained in $A_1 \cup A_2$, which is countable.

Theorem 1.9 If f is continuous on $[a,b]$, then $V = \lim_{\|\rho\| \to 0} V_\rho$; that is, given $M < V$, there exists a $\delta > 0$ s.t. if $\|\rho\| < \delta$, then $V_\rho > M$ for any partition ρ with $\|\rho\| < \delta$.

Pf. Let V be the variation of f over $[a,b]$. Given $M<V$, take an $\varepsilon>0$ s.t. $M+\varepsilon<V$. There exists a partition $\bar{P}=(\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)$ s.t. $V_{\bar{P}}> M+\varepsilon$.

Due to uniform continuity of f over $[a,b]$, for the number $\frac{\varepsilon}{2(m+1)}>0$, there is a δ_1 s.t. if $|x-u|<\delta_1$, then $|f(x)-f(u)|<\frac{\varepsilon}{2(m+1)}$.

Let $\delta<\delta_1, \|\bar{P}\|$, and let P be any partition of $[a,b]$ with $\|P\|<\delta$.

Since $\|P\|<\|\bar{P}\|$, every interval $[x_{i-1},x_i]$ contains at most one $\bar{x}_j$.

Let i_j be s.t. $\bar{x}_j\in[x_{i_j-1},x_{i_j}]$. x_{i_j-1} $\bar{x}_j$ x_{i_j}

$$V_{P\cup\bar{P}}=V_P+\sum_{j=0}^{m}\Big(|f(\bar{x}_j)-f(x_{i_j-1})|+|f(x_{i_j})-f(\bar{x}_j)| - |f(x_{i_j})-f(x_{i_j-1})|\Big)$$

$$\le V_P+\sum_{j=0}^{m}\Big(\underbrace{|f(\bar{x}_j)-f(x_{i_j-1})|}_{\le\frac{\varepsilon}{2(m+1)}}+\underbrace{|f(x_{i_j})-f(\bar{x}_j)|}_{\le\frac{\varepsilon}{2(m+1)}}\Big)$$

since $\bar{x}_j-x_{i_j-1},\ x_{i_j}-\bar{x}_j<\delta<\delta_1$

$\le V_P+\varepsilon$, so

$V_P\ge V_{P\cup\bar{P}}-\varepsilon\ge V_{\bar{P}}-\varepsilon>M$

Thus, if $\|P\|<\delta$, then $V_P>M$. Since M was arbitrary, $V=\lim_{\|P\|\to0}V_P$.

Corollary 1.10 If f has a continuous derivative f' on $[a,b]$, then

$$V=\int_a^b|f'|\qquad P=\int_a^b(f')^+\qquad N=\int_a^b(f')^-$$

Pf. If P is a partition of $[a,b]$, then

$$V_P=\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|=\begin{bmatrix}\text{Mean}\\ \text{Value Thm}\\ \text{on }[x_{i-1},x_i]\end{bmatrix}$$

$$=\sum_{i=1}^{m}|f'(c_i)(x_i-x_{i-1})|=\sum_{i=1}^{m}|f'(c_i)|(x_i-x_{i-1})$$

a Riemann Sum for $\int_a^b|f'|$.

Because $|f'|\in R[a,b]$, we have

$$\int_a^b|f'|=\lim_{\|P\|\to0}S(f,P)$$
$$=\lim_{\|P\|\to0}V_P=V.$$

Now, $P=\frac{1}{2}(V+f(b)-f(a))$

$$=\frac{1}{2}\Big(\int_a^b|f'|+\int_a^b f'\Big)=\frac{1}{2}\int_a^b(|f'|+f')$$
$$=\int_a^b(f')^+$$

$$N=\frac{1}{2}(V-f(b)+f(a))=\frac{1}{2}\Big(\int_a^b|f'|-\int_a^b f'\Big)$$
$$=\int_a^b\frac{1}{2}(|f'|-f')=\int_a^b(f')^-$$

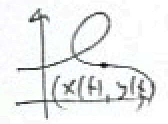
[end of 1.1]

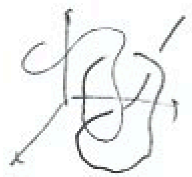
1.2 Rectifiable Curves

A curve in a plane or space is given by two or three parametric equations:

$x = x(t)$ $\quad$ $x = x(t)$
$y = y(t)$ $\quad$ $y = y(t)$
$\qquad\qquad\ \ z = z(t)$

$t \in [a,b]$



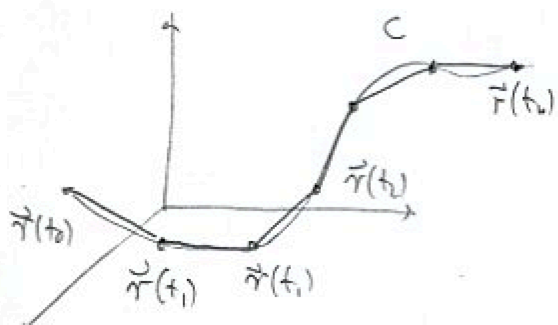


May also view it as a function $\vec{r} : [a,b] \to \mathbb{R}^2 \ (\mathbb{R}^3)$

Def. The graph of $\vec{r}$ is $\{\vec{r}(t) \mid t \in [a,b]\}$

Def. Given a partition $P = (t_0, t_1, \ldots, t_m)$ of $[a,b]$

let $$l_P = \sum_{i=1}^{m} \sqrt{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2}$$



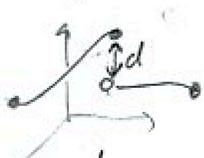
l_P is sum of lengths of line-segments shown

Length of curve C (parametrization, really) is def. as

$$L(C) = \sup \{ l_P \mid P \text{ a part. of } [a,b] \}$$

If $L(C) < \infty$, we call C rectifiable.

Note. 1) If $\vec{r}(t)$ is not continuous, $L(C)$ counts the gaps, too.



$L(C)$ = length of curves + d

2) If $\vec{r}(t)$ traces the graph more than once $L(C)$ takes into account how many times the curve has been traversed.

Theorem 1.13 Let a curve be defined by $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in [a,b]$. Then C is rectifiable if and only if all of x, y, z are rectifiable.

Furthermore,
$$V(x), V(y), V(z) \le L(C) \le V(x) + V(y) + V(z)$$

Pf: $|a|, |b|, |c| \le \sqrt{a^2+b^2+c^2} \le |a|+|b|+|c|$
true by squaring

So

(1) $\sum_{i=1}^{m} |x(t_i)-x(t_{i-1})|, \left|\sum_{i=1}^{m} |y(t_i)-y(t_{i-1})|\right| \le$

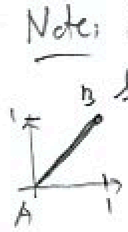
(2) $\sum_{i=1}^{m} \sqrt{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2}$

(3) $\le \sum_{i=1}^{m} |x(t_i)-x(t_{i-1})| + \sum_{i=1}^{m} |y(t_i)-y(t_{i-1})| + \sum_{i=1}^{m} |z(t_i)-z(t_{i-1})|$

If (2) has a finite sup, over all partitions P of $[a,b]$, so does each of the expressions in (1) so x, y, z are of bounded variation

If each item in (1) is of bounded variation, then (3) has a sup that is smaller than the sup of those in (1), hence (2) has a finite supremum.

Note: Let $\vec{r}(t) = (f(t), f(t))$, where $f : [0,1] \to [0,1]$ is of unbounded var. Then $L(C) = \infty$, even though graph lies on the line segment AB



Theorem If each of x, y, z has a continuous derivative on $[a,b]$, then

$$L(C) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

1.3 The Rieman Stieltjes Integral

Def. Let $\dot{P} = (x_0, x_1, \ldots, x_n)$, $t_i \in [x_{i-1}, x_i]$ be a tagged partition of $[a,b]$, $f, \varphi : [a,b] \to \mathbb{R}$ functions. The sum

$$S(f, \dot{P}) = \sum_{i=1}^{n} f(t_i)\big(\varphi(x_i) - \varphi(x_{i-1})\big)$$

is called a Riemann-Stieltjes sum for $\dot{P}$.

If there exists a number L s.t. for every $\varepsilon > 0$, there exists a $\delta > 0$ s.t. if $\|\dot{P}\| < \delta$, then $|S(f, \dot{P}) - L| < \varepsilon$ we say the function is Riemann-Stieltjes integrable and call L the Riemann-Stieltjes integral of f w.r.t. φ on $[a,b]$. Notations:

$$L = \int_a^b f(x)\, d\varphi(x) = \int_a^b f\, d\varphi, \quad f \in RS[a,b].$$

Notes:

1) If $\int_a^b f\, d\varphi$ exists, it is unique (like 7.1.2)

2) $\int_a^b f\, d\varphi$ exists iff for every $\varepsilon > 0$, there exists a $\delta > 0$ s.t. if $\dot{P}, \dot{Q}$ are TP. with $\|\dot{P}\|, \|\dot{Q}\| < \delta$, then $|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon$ (like 7.2.1)

3) If $\varphi(x) = x$, $\int_a^b f\, d\varphi = \int_a^b f$ (Riem. integral is special case of R-S integral)

4) If f, φ' are continuous on $[a,b]$, then

$$\int_a^b f\, d\varphi = \int_a^b f \cdot \varphi'$$

This is because the R-S sum

$$\sum_{i=1}^{n} f(t_i)\big(\varphi(x_i) - \varphi(x_{i-1})\big) = \sum_{i=1}^{n} f(\underbrace{t_i)\, \varphi'(c_i}_{\text{not same}})(x_i - x_{i-1}), \quad c_i \in [x_{i-1}, x_i]$$

nearly a Riem sum for $f \cdot \varphi'$

5) Let φ be a step function, i.e., there is a partition $(a_0, \ldots, a_m)$ s.t. $\varphi|_{(a_{i-1}, a_i)}$ is constant. If we set

$$\varphi(a_i-) = \lim_{x \to a_i-} \varphi(x), \quad \varphi(a_0-) = \varphi(a_0)$$

$$\varphi(a_i+) = \lim_{x \to a_i+} \varphi(x) \quad \varphi(a_m+) = \varphi(a_m)$$

then for a cont. f, $\int_a^b f\, d\varphi = \sum_{i=0}^{m} f(a_i)\big(\varphi(a_i+) - \varphi(a_i-)\big)$

Proof is similar to 7.1.4 b, integral of a step function

6) If f and φ have a common point of discontinuity, then $\int_a^b f\,d\varphi$ doesn't exist.

Suppose f and φ are discontinuous at a $c \in [a,b]$. Then there exist sequences $(u_n), (v_n) \in [a,b]$ and numbers $\varepsilon_1, \varepsilon_2 > 0$ s.t. $u_n, v_n \to c$ and $|f(u_n) - f(v_n)| \geq \varepsilon_1$, $|\varphi(v_n) - \varphi(c)| \geq \varepsilon_2$ for all $n \in \mathbb{N}$. Note: $u_n \neq c$, $v_n \neq c$ $\forall n$.

Case 1 For infinitely many indices n, $u_n > c$ $(u_n < c)$
" " " " " n, $v_n > c$ $(v_n < c)$

(it doesn't have to occur for same indices in u_n and v_n)

Let u_n, v_n be the subsequences with those indices. We still have $u_n \to c$, $v_n \to c$

Given $\delta > 0$, there is a $k_1 \in \mathbb{N}$ s.t. $|c - v_n| < \delta$ for $n \geq k_1$

Then there exists a k_2 s.t. $c \leq u_m \leq v_n$ for $m \geq k_2$. Setting $n = k_1$, $m = k_2$ define TP $\dot{P}$ and $\dot{Q}$, s.t. $\|\dot{P}\|, \|\dot{Q}\| < \delta$:

$P, Q = (a, x_1, \ldots, x_{i-2}, c, v_n, x_{i+1}, \ldots, b)$ and take tags to be same on all subintervals except $[c, v_m]$. There, set $t_i = c$ for $\dot{P}$, $t_i = u_m$ for $\dot{Q}$

Clearly

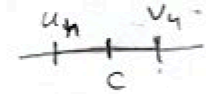
$$|S(f,\dot{P}) - S(f,\dot{Q})|$$
$$= |f(c)(\varphi(v_m) - \varphi(c)) - f(u_m)(\varphi(v_m) - \varphi(c))|$$
$$= |(f(c) - f(u_m))(\varphi(v_m) - \varphi(c))|$$
$$\geq \varepsilon_1 \varepsilon_2 > 0$$

So, for every $\delta > 0$, we have TP $\dot{P}, \dot{Q}$ s.t.

$|S(f,\dot{P}) - S(f,\dot{Q})| \geq$ fixed pos. number,

ruining Cauchy's condition, so f is not R-S integrable.

Case 2 For all indices except finitely many, $u_n < c$, $v_n > c$ $(u_n > c, v_n < c)$.



If $\varphi(u_n) \not\to \varphi(c)$, then we are in a situation like case 1, with $v_n = u_n$.

If $\varphi(u_n) \to \varphi(c)$, let δ be given. Then there is a $k_1 \in \mathbb{N}$ s.t. if $n \geq k_1$, $|u_n - c|, |v_n - c| < \frac{\delta}{2}$. Then is also a $k_2 \in \mathbb{N}$ s.t. if $n \geq k_2$ $|\varphi(u_n) - \varphi(c)| < \frac{\varepsilon_2}{2}$. Let $\dot{P}, \dot{Q}$ be s.t. $\|\dot{P}\|, \|\dot{Q}\| < \delta$,

$P, Q = (a, x_1, \ldots, x_{i-2}, u_n, v_n, x_{i+1}, \ldots, b)$

set $t_i = c$ for $\dot{P}$ and let
$t_i = u_n$ for $\dot{Q}$ $n \geq \max\{k_1, k_2\}$

Then $|S(f,\dot{P}) - S(f,\dot{Q})| = |(f(c) - f(u_n))(\varphi(v_n) - \varphi(u_n))|$

$$= |f(c) - f(u_n)|\,|\varphi(v_n) - \varphi(c) - (\varphi(u_n) - \varphi(c))|$$

$$\geq \varepsilon_1 \big|\underbrace{|\varphi(v_n) - \varphi(c)|}_{\geq \varepsilon_2} - \underbrace{|\varphi(u_n) - \varphi(c)|}_{\leq \frac{\varepsilon_2}{2}}\big| \geq \underbrace{\varepsilon_1\left(\varepsilon_2 - \frac{\varepsilon_2}{2}\right)}_{= \frac{\varepsilon_1 \varepsilon_2}{2}}.$$

Theorem 1.16

1) if $\int_a^b f d\varphi$ exists, so do $\int_a^b c f d\varphi$ and $\int_a^b f\, d(c\varphi)$

and any $\int_a^b c f\, d\varphi = c\int_a^b f\, d\varphi = \int_a^b f\, d(c\varphi)$

2) If $\int_a^b f_1 d\varphi$ and $\int_a^b f_2 d\varphi$ exist, then $\int_a^b f_1+f_2\, d\varphi$ exists and $\int_a^b f_1+f_2\, d\varphi = \int_a^b f_1\, d\varphi + \int_a^b f_2\, d\varphi$

3) If $\int_a^b f d\varphi_1$ and $\int_a^b f d\varphi_2$ exist, then $\int_a^b f d(\varphi_1+\varphi_2)$ exists and $\int_a^b f\, d(\varphi_1+\varphi_2) = \int_a^b f d\varphi_1 + \int_a^b f d\varphi_2$

Pf similar to those for Rieman integrals

Theorem 1.17 If $\int_a^b f d\varphi$ exists and $c\in(a,b)$, then $\int_a^c f d\varphi$ and $\int_c^b f d\varphi$ exist and

$$\int_a^b f d\varphi = \int_a^c f d\varphi + \int_c^b f d\varphi$$

Pf. like for Rieman integrals

Theorem 1.21 If $\int_a^b f d\varphi$ exists, then so does $\int_a^b \varphi\, df$ and vice-versa. $\int_a^b f d\varphi = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi\, df$

(like integration by parts.)

Pf. Let $P=(x_0,\dots,x_n)$, $t_i\in[x_{i-1},x_i]$. Then

$$S(f,\dot P) = \sum_{i=1}^{m} f(t_i)\left(\varphi(x_i)-\varphi(x_{i-1})\right)$$

$$= \sum_{i=1}^{m} f(t_i)\varphi(x_i) - \sum_{i=1}^{n} f(t_i)\varphi(x_{i-1})$$

$$= \sum_{i=1}^{m} f(t_i)\varphi(x_i) - \sum_{i=0}^{m-1} f(t_{i+1})\varphi(x_i)$$

$$= \sum_{i=1}^{m-1}\left(f(t_i)-f(t_{i+1})\right)\varphi(x_i) + f(t_m)\varphi(b) - f(t_1)\varphi(a)$$

$$= -\sum_{i=1}^{m-1}\left(f(t_{i+1})-f(t_i)\right)\varphi(x_i) - \left(f(t_1)-f(a)\right)\varphi(a) - \left(f(b)-f(t_m)\right)\varphi(b) - f(a)\varphi(a) + f(b)\varphi(b)$$

$$= -\underbrace{S(\varphi,\dot Q)}_{} + f(b)\varphi(b) - f(a)\varphi(a)$$

RS-sum for φ wrt f, and tagged partition Q that is a subset of $(a, t_1, t_2, \dots, t_m, b)$ (possibly $t_{i-1}=t_i$ for some i),

Therefore:

A RS-sum for f wrt φ

$= -$(same RS sum for φ wrt f) $+ f(b)\varphi(b) - f(a)\varphi(a)$

(May make same claim for f, φ reversed)

Suppose $\int_a^b \varphi\, df = L$ exists. Then, given ε, there is a δ s.t. $|S(\varphi,Q)-L|<\varepsilon$ for all Q with $\|Q\|<\delta$.

Let P be any partition with $\|P\|<\frac{\delta}{2}$

Then $S(f,\dot P) = -S(f,\dot Q) + f(b)\varphi(b) - f(a)\varphi(a)$

for some TP $\dot Q$, when $\|\dot Q\| < 2\cdot\|\dot P\| < \delta$

but then

$|S(f,\dot Q) - L| < \varepsilon$

so $|S(f,\dot P) + S(f,\dot Q) - S(f,\dot P) - L| < \varepsilon$

$|f(b)\varphi(b) - f(a)\varphi(a) - L - S(f,\dot P)| < \varepsilon$

S.E.w.a, we see that f is RS-integrable wrt. to φ and that

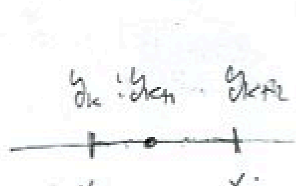
$$\int_a^b f\,d\varphi = f(b)\varphi(b) - f(a)\varphi(a) - L$$

Because the claim is symmetric in f, φ the other direction is proved in the same way, by interchanging f, φ.

When does $\int f\,d\varphi$ exist? Here is a sufficient condition:

<u>Theorem 1.24</u> If f is continuous on $[a,b]$ and φ is of bounded variation on $[a,b]$, then $\int_a^b f\,d\varphi$ exists. Furthermore,

$$\left|\int_a^b f\,d\varphi\right| \le \sup_{x\in[a,b]} |f(x)| \cdot V_{[a,b]}(\varphi)$$



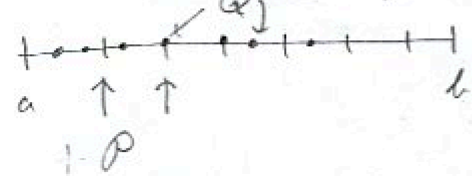
$f(t_i)(\varphi(y_{k+2}) - \varphi(y_k)) = f(t_i)(\varphi(y_{k+1}) - \varphi(y_k) + \varphi(y_{k+2}) - \varphi(y_{k+1}))$

Proof: Let f be cont. and φ of bounded variation $V = V_{[a,b]}(f)$.

Let ε be given. Since f is unif. continuous, for $\frac{\varepsilon}{\ } > 0$ there is a $\delta > 0$ s.t. if $|x-u| < \delta$, then

$|f(x) - f(u)| <$

Now let $\dot P, \dot Q$ be any tagged partitions with $\|\dot P\|, \|\dot Q\| < \delta$, and give the partition $P \cup Q$ any tags.



Clearly, $\|P \cup Q\| < \delta$

Let $P = (x_0, x_1, \dots, x_n)$ with tags t_i

$P \cup Q = (y_0, y_1, \dots, y_m)$ with tags s_j $(m \ge n)$

$$\left|S(f,\dot P) - S(f, \dot{P\cup Q})\right|$$

$$= \left|\sum_{i=1}^{n} f(t_i)(\varphi(x_i) - \varphi(x_{i-1})) - \sum_{j=1}^{m} f(s_j)(\varphi(y_j) - \varphi(y_{j-1}))\right|$$

$=$ [Note that $S(f,\dot P)$ may be written as $\sum_{j=1}^{m} f(u_j)(\varphi(y_j) - \varphi(y_{j-1}))$ ← not a RS-sum] $=$

$\{t_1, \dots, t_n\} = \{u_1, \dots, u_m\}$

t_i's with repetition

$$= \sum_{j=1}^{m} f(u_j)(\varphi(y_j)-\varphi(y_{j-1})) - \sum_{j=1}^{m} f(s_j)(\varphi(y_j)-\varphi(y_{j-1}))$$

$$= \Big| \sum_{j=1}^{m} \underbrace{(f(u_j)-f(s_j))}_{\le \frac{\varepsilon}{}}(\varphi(y_j)-\varphi(y_{j-1}))\Big|$$

since $|u_j - s_j| \le \delta$ as u_j, s_j are both in a subinterval $[x_{i-1}, x_i]$

$$\le \frac{\varepsilon}{2V}\sum_{j=1}^{m}|\varphi(y_j)-\varphi(y_{j-1})| \le \frac{\varepsilon}{2V}\cdot V \le \frac{\varepsilon}{2}$$

Now:

$$|S(f,\dot{P}) - S(f,\dot{Q})|$$

$$\le |S(f,\dot{P}) - S(f,\dot{P}\cup\dot{Q})| + |S(f,\dot{P}\cup\dot{Q}) - S(f,\dot{Q})|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

SEwa, we have satisfied Cauchy's criterion.

Let $L = \int_a^b f\,d\varphi$. Then for any $\varepsilon > 0$ there is a TP $\dot{P}$ s.t. $|L - S(f,\dot{P})| < \varepsilon$

Now: $|L| \le |L - S(f,\dot{P})| + |S(f,\dot{P})|$

$$< \varepsilon + \Big|\sum_{i=1}^{n} f(t_i)(\varphi(x_i)-\varphi(x_{i-1}))\Big|$$

$$\le \varepsilon + \sum_{i=1}^{n} |f(t_i)|\,|\varphi(x_i)-\varphi(x_{i-1})|$$

$$\le \varepsilon + \sum_{i=1}^{n} M\,|\varphi(x_i)-\varphi(x_{i-1})|$$

$\le \varepsilon + MV$, where $M = \sup_{x\in[a,b]}|f(x)|$, $V = V_{[a,b]}(\varphi)$

SEwa, $|L| \le MV$.

Theorem 1.27 (Mean Value Theorem for R-S integrals)

If f is continuous and φ is increasing, then there exists a number $c \in [a,b]$ s.t.

$$\int_a^b f\,d\varphi = f(c)(\varphi(b)-\varphi(a))$$