Mathematical Reasoning — Exam 1
MAT 312, Fall 2015 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If the square of an integer is greater than one, then the integer is greater than one.

2. (2pts) The Earth is round or the moon is closer than 1,000 miles away.

3. (3pts) (universal set=$\mathbf{R}$) $3x - 1 = 3(x + 2) - 7$

4. (3pts) (universal set=$\mathbf{Z}$) $|x| \leq 10$ and x is divisible by 4.

5. (4pts) For every $x \in \mathbf{R}$, $x^2 + 6x + 11 > 0$.

Negate the following statements.

6. (3pts) President Xi supports building a Disney resort in China or puts an order for 300 Boeing airplanes.

7. (3pts) If the Pope says to look after the poor, then I will volunteer for a charity.

8. (6pts) Use a truth table to prove that $(P \implies Q) \vee (Q \implies P)$ is a tautology. (Use however many columns you need.)

P	Q							
T	T							
T	F							
F	T							
F	F							

9. (12pts) Use previously proven logical equivalences to prove the equivalence $P \vee (Q \implies R) \equiv (Q \implies P) \vee R$. Do not use a truth table.

10. (4pts) Write the converse and contrapositive of the statement: if an integer is divisible by 6, then it is divisible by 3.

Converse:

Contrapositive:

11. (10pts) Suppose the following statements are true:
If Benny gets a ball, then he doesn't get a bicycle.
If Benny gets a bicycle, then he gets a helmet.
Benny didn't get a helmet and he didn't get a ball.

Determine truth value of the following statements and justify.

Benny got a ball.

Benny got a bicycle.

12. (4pts) Use set builder notation to write the set $\{-1, 8, -27, 64, -125, \dots\}$.

13. (7pts) A function $f : (a, b) \to \mathbf{R}$ is said to be differentiable on (a, b) if $f'(c)$ exists for every c in (a, b).
a) Write the definition using symbols.
b) Negate the definition using symbols.
c) Finish the sentence: "A function f is not differentiable on (a, b) if ..."

14. (10pts) For every natural number m, there exists a natural number n such that mn is a square of a natural number.
a) Write this statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(x + y^2 \geq 7)$$

a) If $x = 3$, is the statement true?
b) If $x = 8$, is the statement true?
c) Find the truth set (the x's) of the above statement.

16. (15pts) We will call an integer n type-0, type-1, type-2,..., or type-9 if it can be written in the form $n = 10k$, $n = 10k + 1$, $n = 10k + 2, \ldots$, or $n = 10k + 9$, respectively, for some integer k. Use this idea to show the following: if an integer m ends with 4, and an integer n ends with 7, then the integer $m^2 + mn + n$ ends with 1. Start with a know-show table if you find it helpful.

Bonus. (10pts) Show by example that quantifiers are not "commutative." That is, give an example of an open sentence $P(x, y)$ about real numbers x and y, so that the statements below have opposite truth values (justify why they do).

$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})P(x, y)$ $\qquad\qquad$ $(\exists y \in \mathbf{R})(\forall x \in \mathbf{R})P(x, y)$

Mathematical Reasoning — Exam 2
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Show all your work!

1. (10pts) Prove the transitive property for congruences (n is a natural number): for all integers a, b and c, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$.

2. (16pts) Prove using induction: for every natural number n, $1+3+5+7+\cdots+(2n-1) = n^2$.

3. (12pts) Let p be a rational number. Prove: for every real number x, if x is irrational, then $\dfrac{1}{p+x}$ is irrational.

4. (22pts) Consider the statement: for every integer n, n is divisible by 5 if and only if n^2+n is divisible by 5.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

5. (20pts) Prove the following:
a) For every integer a, if a^2 is divisible by 6, then a is divisible by 6.
b) $\sqrt{6}$ is an irrational number. (Use statement a)).

6. (6pts) Use the triangle inequality to prove that for all real numbers c, d,

$|c+1-(d+3)| < |c-d|+2.$

7. (14pts) Prove that for all real numbers a, b, $b \neq 0$, $\dfrac{2a}{b} \leq a^2 + \dfrac{1}{b^2}$.

Bonus. (10pts) Use the facts that $\sqrt{2}$ is irrational and that $0 < \frac{\sqrt{2}}{2} < 1$ to show that between any two rational numbers a and b there exists an irrational number.

Mathematical Reasoning — Exam 3
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Show all your work!

1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A \cap B) - C$ $\qquad$ $C - (A \cap B)$ $\qquad$ $(C - A) \cup (C - B)$ $\qquad$ $(A \cup B) \cap C$

2. (6pts) Draw arrow diagrams between two sets that illustrate

a) a bijection $\quad$ b) a surjection that is not an injection $\quad$ c) an f where range $f \neq$ codom f

3. (12pts) Let U be the set of integers. Consider the sets $A = \{k \in \mathbf{Z} \mid k \equiv 3 \pmod 5\}$, $B = \{k \in \mathbf{Z} \mid k \text{ is even}\}$, $C = \{k \in \mathbf{Z} \mid -20 \leq k \leq 20\}$ and write the following subsets using the roster method (pattern needs to be obvious).

$A \cap B$ $\qquad$ $A - B$ $\qquad$ B^c $\qquad$ $A \cap (B \cup C)$ $\qquad$ $C - (A \cup B)$ $\qquad$ $B - A$

4. (14pts) Let $A = \{n \in \mathbf{N} \mid n \text{ is a sum of three consecutive natural numbers}\}$ and $B = \{n \in \mathbf{N} \mid n \text{ is divisible by } 3)\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

5. (12pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ be given by $f(m, n) = 2m - 3n$.
a) Evaluate $f(0, 7)$ and $f(1, -3)$.
b) Determine the set of preimages of 4. List at least three elements of this set and illustrate it in the plane.

6. (16pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $g : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = g(x) = 3x + 7 \pmod 5$. Note that f and g have the same formula, but different domains.
a) Write the table of function values for f.
b) Calculate $g(8)$, $g(-4)$ and $g(100)$.
c) What is the set of preimages of 3 under f?
d) What is the set of preimages of 3 under g? Justify.
e) Is f injective? Justify.
f) Is g injective? Justify.

7. (12pts) Let $f(x) = \dfrac{2x}{x+5}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

8. (14pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

Bonus. (10pts) Let S be the set of all functions $f : (0, 1) \to \mathbf{R}$ that are differentiable on $(0, 1)$, and let T be the set of all functions $g : (0, 1) \to \mathbf{R}$. Let $D : S \to T$ be the function of differentiation, that is, $D(f) = f'$.
a) If $f(x) = x^2 - 3x$, find $D(f)$.
b) What is the set of preimages of g, $g(x) = x^3 - 7x$?
c) What is the set of preimages of h, $h(x) = 1$ for $x \in \left(0, \frac{1}{2}\right]$, and $h(x) = -1$ for $x \in \left(\frac{1}{2}, 0\right)$?