

1. (12pts) Are the following statements true or false? Justify.

a) If an integer is divisible by 6, then it is divisible by 3.

b) If roses are blue, then violets are red.

c) For every real number x , $\sqrt{x^2} = x$.

2. (10pts) Negate the following statements.

a) The pillow is blue and fluffy.

b) If $x > 4$, then $x^2 + 2x - 4 < 0$.

c) There exists an $x \in \mathbf{R}$ such that $x^3 + x^2 + x < 0$.

d) For every $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ such that $x < n$.

3. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \vee Q) \implies R \equiv (P \implies R) \wedge (Q \implies R)$. Do not use a truth table.

4. (12pts) Let $\mathbf{Z}_{10} = \{0, 1, 2, \dots, 8, 9\}$ be the universal set. Let $A = \{0, 1, 4, 9\}$, $B = \{x \in \mathbf{Z}_{10} \mid x \equiv 0 \pmod{3}\}$, $C = \{x \in \mathbf{Z}_{10} \mid x \equiv 1 \pmod{2}\}$. Use the roster method to write the following sets:

$A \cap C$ $A \cup C$ $A - B$ $B - C$ B^c $A \cap B \cap C$

5. (12pts) Let A , B and C be subsets of some universal set U .

a) Use Venn diagrams to draw the following subsets.

b) Among the three sets, two are equal. Use set algebra to show they are equal.

$A - (B \cap C)$ $(A - C) \cap B$ $(A \cap B) - C$

6. (14pts) Prove that an integer n is divisible by 5 if and only if $n^2 - 5n$ is divisible by 5.

7. (10pts) Let x be a real number, and p and q rational numbers. Prove: if x is irrational, then $p + \frac{q}{x}$ is irrational.

8. (16pts) Let a and b be integers. Prove or disprove the following statements.

a) If ab is divisible by 4, then a is divisible by 4 or b is divisible by 4.

b) If ab is divisible by 3, then a is divisible by 3 or b is divisible by 3.

9. (12pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \rightarrow \mathbf{Z}_5$, $f(x) = 4x \pmod{5}$.

a) Write the table of function values.

b) What is the set of preimages of 2?

c) Is f injective? Justify.

d) Is f surjective? Justify.

10. (16pts) Let $h(x) = \frac{3x - 2}{4x + 1}$.

- a) What subset of real numbers is the natural domain for this function?
- b) Show that this function is injective.
- c) What is the range of this function? Justify your answer.
- d) What should be the domain and codomain if we want h to be a bijection?

11. (14pts) Use induction to prove: for every $n \in \mathbf{N}$, $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$.

12. (10pts) Let $g : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} \times \mathbf{Z}$ be given by $g(m, n) = (-2m + 9n, m - 5n)$. Show that g is surjective.

Bonus. (7pts) Let $x, y \geq 0$ be real numbers. Show that $\sqrt{xy} \leq \frac{x + y}{2}$.

Bonus. (8pts) Let A, B be subsets of a universal set U . Prove: if $A - B = B - A$, then $A = B$.