Mathematical Reasoning — Final Exam
MAT 312, Fall 2015 — D. Ivanšić

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Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (3pts) If an integer is greater than 3, then it is less than 101.

False, $k=102$ is greater than 3 and is greater than 101

2. (3pts) For every x ($x \in \mathbb{R}$), if $x^2 < -1$, then $3x - 1 = 4x$.

True. The statement $x^2 < -1$ is always false, making the if-then statement true.

3. (3pts) (universal set=**R**) $\sqrt{x^2} = x$.

Open sentence. $\sqrt{x^2} = |x|$ and $|x| = x$ if and only if $x \geq 0$

Truth set: $[0, \infty)$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If n is an integer divisible by 4, then its sum of digits is divisible by 4.

n is divisible by 4 and its sum of digits is not divisible by 4.

5. (4pts) For every natural number m, there exists a natural number n such that $m - n$ is a square of a natural number.

$(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(m-n \text{ is a square}) \rightarrow (\exists m \in \mathbb{N})(\forall n \in \mathbb{N})(m-n \text{ is not a square})$

There exists a natural number m such that for every natural number n, $m-n$ is not a square of a natural number.

6. (4pts) Write the converse and contrapositive of the statement: if n is divisible by 7, then n^2 is divisible by 7.

Converse: If n^2 is divisible by 7, then n is divisible by 7

Contrapositive: If n^2 is not divisible by 7, then n is not divisible by 7.

7. (10pts) Use previously proven logical equivalences to prove the equivalence $P \Longrightarrow (Q \Longrightarrow R) \equiv (P \wedge Q) \Longrightarrow R$. Do not use a truth table.

$$P \Rightarrow (Q \Rightarrow R) \equiv P \Rightarrow (\neg Q \vee R) \equiv \neg P \vee (\neg Q \vee R)$$
$$\equiv (\neg P \vee \neg Q) \vee R \equiv \neg(P \wedge Q) \vee R$$
$$\equiv (P \wedge Q) \Rightarrow R$$

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(3x + y^2 = 4)$$

a) If $x = -1$, is the statement true?
b) If $x = 3$, is the statement true?
c) Find the truth set (the x's) of the above statement.

a) $x=-1$

$(\exists y \in \mathbb{R})(-3+y^2=4)$

$y^2=7$

$y=\pm 7$

True, $-3+y^2=4$ has a solution in $\mathbb{R}$

b) $x=3$

$(\exists y \in \mathbb{R})(9+y^2=4)$

$y^2=-5$

no solution

False: $9+y^2=4$ has no solution in $\mathbb{R}$.

c) $3x+y^2=4$

$y^2=4-3x$

In order for this equation to have a real solution y, $4-3x$ has to be ≥ 0,

$4-3x \geq 0$

$3x \leq 4$

$x \leq \frac{4}{3}$

$(-\infty, \frac{4}{3}]$ truth set

9. (10pts) Let $p \neq 0$ be a rational number. Prove: for every real number x, if x is irrational, then $\dfrac{p}{1+x}$ is irrational.

We prove the contrapositive; If $\frac{p}{1+x}$ is rational, then x is rational,

Suppose $\frac{p}{1+x} = q$, where q is rational. Then $\frac{1+x}{p} = \frac{1}{q}$

$1+x = \frac{p}{q}$, and $x = \frac{p}{q} - 1$, which is a rational number, since

p, q are rational numbers,

10. (14pts) Prove using induction: for every $n \in \mathbf{N}$, $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(3 - \frac{1}{3^n}\right)$.

Base, $n=1$: $1 + \frac{1}{3} = \frac{4}{3}$ $\Big\}$ equal, so true for $n=1$

$\frac{1}{2}\left(3-\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$

Step: suppose statement is true for $n=k$; $1+\frac{1}{3}+\dots+\frac{1}{3^k} = \frac{1}{2}\left(3-\frac{1}{3^k}\right) \Big| + \frac{1}{3^{k+1}}$

$$1+\frac{1}{3}+\dots+\frac{1}{3^k}+\frac{1}{3^{k+1}} = \frac{1}{2}\left(3-\frac{1}{3^k}\right)+\frac{1}{3^{k+1}} = \frac{3}{2} - \frac{1}{2\cdot 3^k} + \frac{1}{3^{k+1}}$$

$$= \frac{3}{2} + \frac{-3+2}{2\cdot 3^{k+1}} = \frac{3}{2} - \frac{1}{2\cdot 3^{k+1}} = \frac{1}{2}\left(3 - \frac{1}{3^{k+1}}\right)$$ which is the statement for $n = k+1$.

11. (14pts) Consider the statement: for every integer n, n is divisible by 4 if and only if $2n^2 + 5n$ is divisible by 4.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

a) For every integer n, if n is divisible by 4, then $2n^2+5n$ is divisible by 4 and if $2n^2+5n$ is divisible by 4, then n is divisible by 4.

b) $\Rightarrow$) Suppose n is divisible by 4. Then there is an integer k s.t. $n = 4k$. Then $2n^2+5n = 2(4k)^2 + 5\cdot 4k = 32k^2+20k = 4(8k^2+5k)$, which is divisible by 4.

$\Leftarrow$)

r	$2r^2+5r$ (mod 4)
0	0
1	3
2	2
3	1

We prove the contrapositive: If n is not divisible by 4, then $2n^2+5n$ is not divisible by 4.

Suppose n is not divisible by 4. Then $n \equiv r \pmod 4$ where $r = 1, 2, 3$, and $2n^2+5n \equiv 2r^2+5r \pmod 4$. Since for $r = 1, 2, 3$, $2r^2+5r$ is never congruent to $0 \pmod 4$, we get that $2n^2+5n$ is not divisible by 4.

c) Both cond. statements are true, so statement is true.

12. (10pts) Prove that for every real number a, if $a > -3$, then $a + 1 + \frac{1}{a+3} \geq 0$.

Exploration:

$a+1+\frac{1}{a+3} \geq 0 \quad | \cdot a+3$

$(a+1)(a+3)+1 \geq 0$

$a^2+4a+3+1 \geq 0$

$a^2+4a+4 \geq 0$

$(a+2)^2 \geq 0$ is true

Proof: Let $a > -3$. Then $a+3 > 0$

$(a+2)^2 \geq 0$ for every $a \in \mathbb{R}$

$a^2+4a+4 \geq 0$

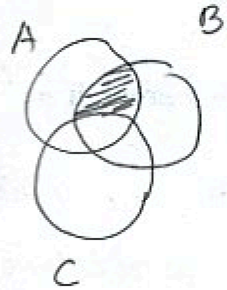
$a^2+4a+3+1 \geq 0$

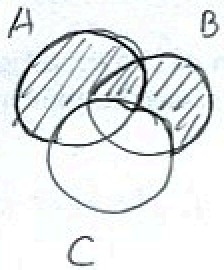
$(a+1)(a+3)+1 \geq 0 \quad | \div (a+3)$, a pos. no.

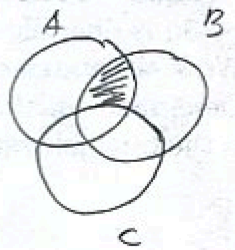
$a+1+\frac{1}{a+3} \geq 0$

13. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$(A \cap B) - (A \cap C)$ | $(A - C) \cup (B - C)$ | $(A \cap B) - C$







1st and 3rd are equal: $(A \cap B) - (A \cap C) = A \cap B \cap (A \cap C)^c$

$= (A \cap B) \cap (A^c \cup C^c) = ((A \cap B) \cap A^c) \cup ((A \cap B) \cap C^c)$

$= ((A \cap A^c) \cap B) \cup (A \cap B - C) = (A \cap B) - C$

$\emptyset$

14. (12pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 5 \pmod 6\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv 2 \pmod 3\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

$A = \{\ldots, -7, -1, 5, 11, 17, \ldots\}$ $B = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$

a) $A \subseteq B$. Let $k \in A$. Then $k-5 = 6q$ for some $q \in \mathbb{Z}$ so $k-2 = 6q+3$

$k-2 = 3(2q+1)$ so $k \equiv 2 \pmod 3$, that is, $k \in B$

b) $2 \equiv 2 \pmod 3$, yet $2 \not\equiv 5 \pmod 6$

$(5-2 = 3$, not a multiple of $6)$

so $2 \in B$ and $2 \notin A$, hence $B \not\subseteq A$.

15. (14pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ be given by $f(m,n) = 2m - 3n$.
a) Evaluate $f(0,7)$ and $f(1,-3)$.
b) Determine the set of preimages of 1. List at least three elements of this set and illustrate it in the plane.
c) Is this function injective?
d) Is this function surjective? *(Hint: if $2m - 3n = 1$ has a solution, it's easy to find the solution of $2m - 3n = k$. How?)*

a) $f(0,7) = -21$

$f(1,-3) = 11$

b) $f(m,n) = 1$

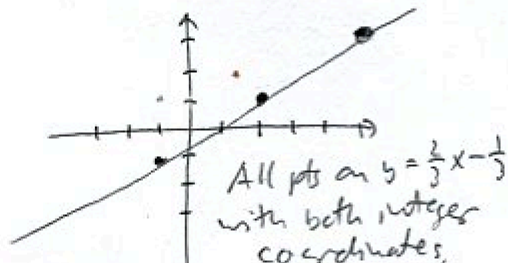
$2m - 3n = 1$

$\{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m - 3n = 1\}$

contains $(2,1), (5,3), (-1,-1), \ldots$

$2x - 3y = 1$

$y = \frac{2}{3}x - \frac{1}{3}$



c) Function is clearly not injective:

$(2,1) \neq (-1,-1)$, yet

$f(2,1) = 1 = f(-1,-1)$

d) We know that

$f(2,1) = 1$

$2 \cdot 2 - 3 \cdot 1 = 1 \quad | \cdot k$

$2 \cdot 2k - 3 \cdot k = k$

The equation $2m - 3n = k$ has the at least the solution $(2k, k)$, so f is surjective

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

$\Rightarrow$) Prove by contradiction: suppose $A \subseteq B$ and $A \cap B^c \neq \emptyset$.
Then there exists an $x \in A \cap B^c$ so $x \in A$ and $x \in B^c$, that is, $x \in A$ and $x \notin B$. However, since $x \in A$, then $x \in B$, contradicting $x \notin B$.

$\Leftarrow$) Prove by contradiction: suppose $A \cap B^c = \emptyset$ and $A \nsubseteq B$. Then there exists an element $x \in A$ s.t. $x \notin B$, that is $x \in A$ and $x \in B^c$ so $x \in A \cap B^c$, contradicting $A \cap B^c = \emptyset$.

17. (10pts) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2 - 5x + 3$. Determine the range of this function algebraically.

Need $\{y \mid x^2 - 5x + 3 = y$ has a real solution for $x\}$

$x^2 - 5x + 3 = y$ solve for x

$x^2 - 5x + 3 - y = 0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (3-y)}}{2 \cdot 1}$$

$$x = \frac{5 \pm \sqrt{25 + 12y}}{2}$$

has a real solution iff

$25 + 12y \geq 0$

$y \geq -\frac{25}{12}$. So range $= \{y \mid y \geq -\frac{25}{12}\}$

$= [-\frac{25}{12}, \infty)$

Bonus 1. (8pts) Show by example that quantifiers are not "commutative." That is, give an example of an open sentence $P(x, y)$ about real numbers x and y, so that the statements below have opposite truth values (justify why they do).

$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})P(x, y)$ $\quad P(x,y) =$ "$x + y = 5$" $\quad (\exists y \in \mathbf{R})(\forall x \in \mathbf{R})P(x, y)$

$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 5)$

True: for every x, the equation $x + y = 5$ has a solution in y, namely $y = 5 - x$

$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = 5)$

False: Given y,

for $x_1 = 5 - y$ we have $x_1 + y = 5$

for $x_2 = 4 - y$ we have $x_2 + y = 4$,

so $x + y$ does not equal 5 for all x.

Bonus 2. (7pts) If x and y are irrational and $y \neq \dfrac{1}{x}$, does it follow that xy is irrational? *(Hint: difference of squares.)*

$x = \sqrt{2} + 2$ and $y = \sqrt{2} - 2$ are both irrational, yet $(\sqrt{2} - 2)(\sqrt{2} + 2) = 2 - 4 = -2$ is rational

since their product $xy \neq 1$, $y \neq \frac{1}{x}$