Mathematical Reasoning — Exam 3
MAT 312, Fall 2015 — D. Ivanšić

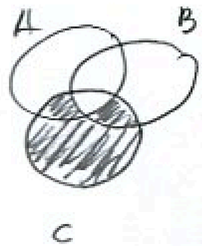
Name: Saul Ocean

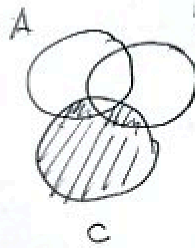
Show all your work!

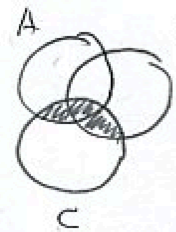
1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A\cap B)-C$ $\qquad$ $C-(A\cap B)$ $\qquad$ $(C-A)\cup(C-B)$ $\qquad$ $(A\cup B)\cap C$

a)







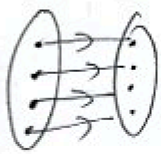
are equal

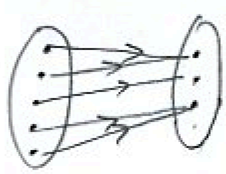
b) $C-(A\cap B) = C\cap(A\cap B)^c = C\cap(A^c\cup B^c) = (C\cap A^c)\cup(C\cap B^c)$

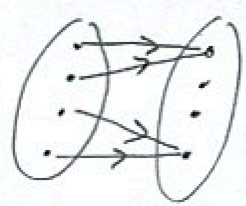
$= (C-A)\cup(C-B)$

2. (6pts) Draw arrow diagrams between two sets that illustrate

a) a bijection $\quad$ b) a surjection that is not an injection $\quad$ c) an f where range $f \neq$ codom f







3. (12pts) Let U be the set of integers. Consider the sets $A=\{k\in\mathbf{Z}\mid k\equiv 3 \pmod 5\}$, $B=\{k\in\mathbf{Z}\mid k \text{ is even}\}$, $C=\{k\in\mathbf{Z}\mid -20\le k\le 20\}$ and write the following subsets using the roster method (pattern needs to be obvious).

$A\cap B$ $\qquad$ $A-B$ $\qquad$ B^c $\qquad$ $A\cap(B\cup C)$ $\qquad$ $C-(A\cup B)$ $\qquad$ $B-A$

$\{\ldots,-12,-2,8,18,\ldots\}$

$\{-17,-7,3,13,23,\ldots\}$

$\{\ldots,-5,-3,-1,1,3,5,\ldots\}$

$\{-32,-22,-17,-12,-7,-2,3,8,13,18,28,38,\ldots\}$

$\{-19,-15,-13,-9,-5,-3,-1,1,5,7,9,11,15,17,19\}$

$\{\ldots,-14,-10,-8,-6,-4,0,2,4,6,10,12,14,16,\ldots\}$

$k-3=5q$, $k=5q+3$, $A=\{-12,-7,-2,3,8,13,18\ldots\}$ $\quad$ $B=\{-6,-4,-2,0,2,4,6,\ldots\}$ $\quad$ $C=\{-20,-19,\ldots,19,20\}$

4. (14pts) Let $A = \{n \in \mathbf{N} \mid n$ is a sum of three consecutive natural numbers$\}$ and $B = \{n \in \mathbf{N} \mid n$ is divisible by 3)$\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

a) Let $n \in A$. Then $n = k + (k+1) + (k+2)$ for some $k \geq 1$, so $n = 3k+3 = 3(k+1)$, hence $n \in \mathbb{N}$ and n is divisible by 3

b) $3 \in B$ but 3 is not a sum of three consecutive natural numbers, which is always ≥ 6
(If $k \geq 1$, $k + k+1 + k+2 = 3k+3 \geq 6$)

($A = \{6, 9, 12, 15, \ldots\}$ $B = \{3, 6, 9, 12, \ldots\}$

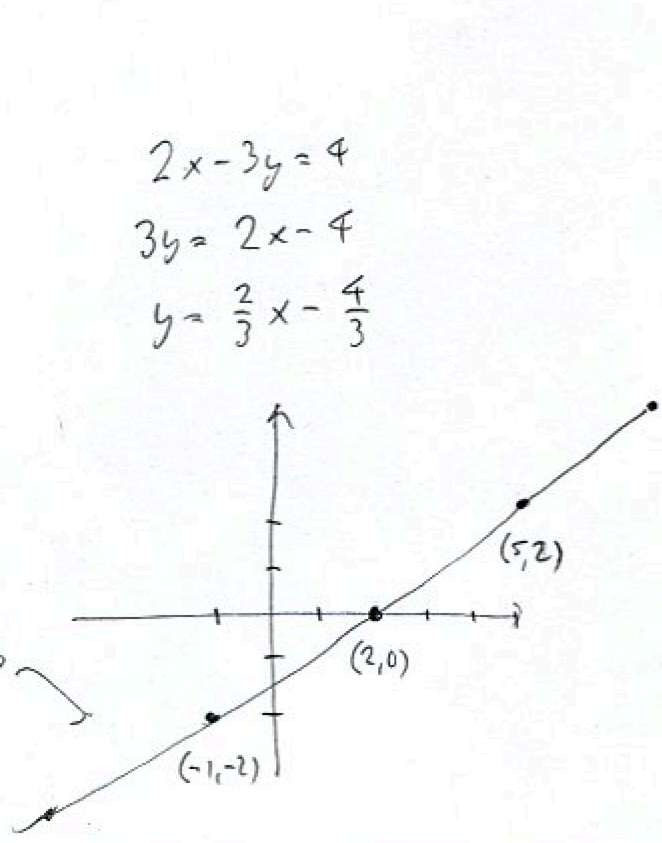
5. (12pts) Let $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ be given by $f(m, n) = 2m - 3n$.
a) Evaluate $f(0, 7)$ and $f(1, -3)$.
b) Determine the set of preimages of 4. List at least three elements of this set and illustrate it in the plane.

a) $f(0,7) = 2 \cdot 0 - 3 \cdot 7 = -21$
$f(1,-3) = 2 \cdot 1 - 3(-3) = 11$

$2x - 3y = 4$
$3y = 2x - 4$
$y = \frac{2}{3}x - \frac{4}{3}$

b) Preimages of 4
$= \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m - 3n = 4\}$
$= \{\ldots, (2,0), (-1,-2), (5,2), \ldots\}$

all points on line $y = \frac{2}{3}x - 4$ whose both coordinates are integers



6. (16pts) Let $\mathbf{Z}_5 = \{0,1,2,3,4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $g : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = g(x) = 3x+7 \pmod 5$. Note that f and g have the same formula, but different domains.
a) Write the table of function values for f.
b) Calculate $g(8)$, $g(-4)$ and $g(100)$.
c) What is the set of preimages of 3 under f?
d) What is the set of preimages of 3 under g? Justify.
e) Is f injective? Justify.
f) Is g injective? Justify.

a)

x	$3x+7 \pmod 5$
0	2
1	0
2	3
3	1
4	4

b)

x	$3x+7 \pmod 5$
8	1
-4	0
100	2

c) preimages of $3 = \{2\}$ (from table)

d) If $x \equiv 2 \pmod 5$

then $3x+7 \equiv 13 \pmod 5 \equiv 3 \pmod 5$

so $g(x) = 3$

If $x \equiv 0,1,3,4 \pmod 5$ then

$3x+7 \equiv 2,0,1,4 \pmod 5$, so $g(x) \neq 3$.

Therefore preimages of $3 = \{x \mid x \equiv 2 \pmod 5\}$
$= \{\ldots, -13, -8, -3, 2, 7, 12, \ldots\}$

e) Yes, table shows f is one-to-one.

f) g is not injective, for example, $8 \neq 3$, yet $g(8) = g(3) = 1$

7. (12pts) Let $f(x) = \dfrac{2x}{x+5}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

a) Can't have
$x+5=0$
$x=-5$

Domain $= \{x \mid x \neq -5\}$
$= (-\infty,-5) \cup (5,\infty)$

b) For which $y \in \mathbb{R}$ can we solve
$\dfrac{2x}{x+5} = y$ for x?

$2x = y(x+5)$

$2x = yx + 5y$

$2x - yx = 5y$

$x(2-y) = 5y$

$x = \dfrac{5y}{2-y}$

can be solved whenever $y \neq 2$

range $f = \{y \mid y \neq 2\}$
$= (-\infty, 2) \cup (2, \infty)$

8. (14pts) Let A, B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

$\Rightarrow$) Let $A \subseteq B$. We show by contradiction that $A \cap B^c = \emptyset$.
Suppose $x \in A \cap B^c$. Then $x \in A$ and $x \in B^c$. Since $x \in A$,
$A \subseteq B$ tells us that $x \in B$. But this contradicts $x \in B^c$,
so $A \cap B^c = \emptyset$

$\Leftarrow$) Suppose $A \cap B^c = \emptyset$. We show $A \subseteq B$ by contradiction.
Suppose there is an x s.t. $x \in A$ and $x \notin B$. Then $x \in A \cap B^c$,
but this contradicts $A \cap B^c = \emptyset$. Hence $A \subseteq B$.

Bonus. (10pts) Let S be the set of all functions $f : (0,1) \to \mathbf{R}$ that are differentiable on $(0,1)$, and let T be the set of all functions $g : (0,1) \to \mathbf{R}$. Let $D : S \to T$ be the function of differentiation, that is, $D(f) = f'$.
a) If $f(x) = x^2 - 3x$, find $D(f)$.
b) What is the set of preimages of g, $g(x) = x^3 - 7x$?
c) What is the set of preimages of h, $h(x) = 1$ for $x \in (0, \frac{1}{2}]$, and $h(x) = -1$ for $x \in (\frac{1}{2}, 0)$?

a) $D(f) = 2x - 3$

b) $D(f) = g$ says

$f' = x^3 - 7x$

$f(x) = \frac{x^4}{4} - \frac{7x^2}{2} + C$

Preimages of g is the set

$\left\{ f : (0,1) \to \mathbb{R} \mid f(x) = \frac{x^4}{4} - \frac{7x^2}{2} + C \right\}$

c) Let $f : (0,1) \to \mathbb{R}$ be s.t. $D(f) = h$

$f'(x) = h(x)$

So For:

$x \in (0, \frac{1}{2}]$ $x \in (\frac{1}{2}, 1)$

$f'(x) = 1$ $f'(x) = -1$

$f(x) = x + C$ $f(x) = -x + d$

Since f needs to be diff., it has to be cont., so $\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}+} f(x)$

$\frac{1}{2} + C = -\frac{1}{2} + d$

$d = C + 1$

But any $f(x) = \begin{cases} x + C, & x \in (0, \frac{1}{2}] \\ -x + C + 1, & x \in (\frac{1}{2}, 1) \end{cases}$
has a sharp point, so is not diff. Hence, set of preimages of h is empty