Mathematical Reasoning — Exam 2 MAT 312, Fall 2015 — D. Ivanšić

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Show all your work!

1. (10pts) Prove the transitive property for congruences (n is a natural number): for all integers a, b and c, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$.

Let $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. Then there exist integers q_1 and q_2 such that

$$a - b = q_1 n \qquad \text{adding equations, we get}$$
$$b - c = q_2 n$$

$$a - c = n(q_1 + q_2)$$, hence $n \mid a-c$, so $a \equiv c \pmod{n}$

2. (16pts) Prove using induction: for every natural number n, $1+3+5+7+\cdots+(2n-1) = n^2$.

We prove statement using mathematical induction.

Base step: $1 = 1^2$ is correct

Suppose statement is true for $n=k$, that is

$$1+3+5+7+\cdots+(2k-1) = k^2 \quad \Big|\ +2k+1$$

$$1+3+5+\cdots+(2k-1)+2k+1 = k^2+2k+1$$

$$1+3+5+\cdots+2k-1+\overbrace{2(k+1)-1} = (k+1)^2$$

which is exactly the statement for $n=k+1$

3. (12pts) Let p be a rational number. Prove: for every real number x, if x is irrational, then $\frac{1}{p+x}$ is irrational.

We prove the contrapositive: if $\frac{1}{p+x}$ is rational, then x is rational,

Suppose $\frac{1}{p+x} = q$, q a rational number. Since $1 \neq 0$, $q \neq 0$, so

we may take reciprocals: $p+x = \frac{1}{q}$, so $x = \frac{1}{q} - p$.

By closure of $\mathbb{Q}$ with respect to $-, \div$ we get that $\frac{1}{q} - p \in \mathbb{Q}$,

hence x is a rational number,

4. (22pts) Consider the statement: for every integer n, n is divisible by 5 if and only if $n^2 + n$ is divisible by 5.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

a) For every integer n, if $5 | n$, then $5 | n^2+n$ and if $5 | n^2+n$, then $5 | n$.

b) ($\Rightarrow$) If $5 | n$, then $5 | n^2+n$: If $5 | n$, then $n = 5k$ for some integer k.

Then $n^2+n = (5k)^2 + 5k = 25k^2 + 5k = 5(\underline{5k^2+k})$. Since $5k^2+k$ is an integer,

$5 | n^2+n$.

($\Leftarrow$) If $5 | n^2+n$, then $5 | n$. We try to prove the contrapos: if $5 \nmid n$, then $5 \nmid n^2+n$.

Let $r = 0, 1, 2, 3, 4$ s.t. $n^2+n \equiv r \pmod 5$. Then a congruence

computation gives the table

r	$n^2+n \equiv \square \bmod 5$
0	0
1	2
2	1
3	2
4	0

which suggests the counterexample $n = 4$

$5 | 4^2+4$ but $5 \nmid 4$.

c) The biconditional statement is not true, since one of the conditional statements is not true

7. (14pts) Prove that for all real numbers a, b, $b \neq 0$, $\frac{2a}{b} \leq a^2 + \frac{1}{b^2}$.

Explanation:

$\frac{2a}{b} \leq a^2 + \frac{1}{b^2} \quad | \cdot b^2$

$2ab \leq a^2b^2 + 1$

$0 \leq a^2b^2 - 2ab + 1$

$0 \leq (ab-1)^2$

which is true

Proof: let $a, b \in \mathbb{R}$, $b \neq 0$

Since a square is always ≥ 0,

$(ab-1)^2 \geq 0$

$a^2b^2 - 2ab + 1 \geq 0$

$a^2b^2 + 1 \geq 2ab \quad | \cdot \frac{1}{b^2}$, a positive number

$a^2\cancel{b^2} \cdot \frac{1}{\cancel{b^2}} + \frac{1}{b^2} \geq 2a\cancel{b} \cdot \frac{1}{b^{\cancel{2}}}$

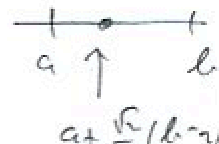
$a^2 + \frac{1}{b^2} \geq \frac{2a}{b}$

Bonus. (10pts) Use the facts that $\sqrt{2}$ is irrational and that $0 < \frac{\sqrt{2}}{2} < 1$ to show that between any two rational numbers a and b there exists an irrational number.

Without loss of generality, let $a < b$. Then $b - a > 0$

We have, $0 < \frac{\sqrt{2}}{2} < 1 \quad | \cdot (b-a)$

$0 < \frac{\sqrt{2}}{2}(b-a) < b - a \quad | + a$



$a < a + \frac{\sqrt{2}}{2}(b-a) < b$

Now $a + \frac{b-a}{2}\sqrt{2}$ has form $p + q\sqrt{2}$, with p, q rational, so is irrational, (Fact from class)

(If $p + q\sqrt{2} = r$ were rational, then $\sqrt{2} = \frac{r-p}{q}$ would be rational)

5. (20pts) Prove the following:
a) For every integer a, if a^2 is divisible by 6, then a is divisible by 6.
b) $\sqrt{6}$ is an irrational number. (Use statement a)).

a) We prove the contrapositive:
if a is not divisible by 6, then a^2 is not divisible by 6:
We know that $a \equiv 1, 2, .. 5 \pmod{6}$, so we get that
$a^2 \equiv 1, 3, 4 \pmod{6}$. Since $a^2 \not\equiv 0 \pmod{6}$, a^2 is not divisible by 6.

$a \equiv \square$	$a^2 \equiv \square \pmod{6}$
1	1
2	4
3	3
4	4
5	1

b) We prove by contradiction. Suppose $\sqrt{6}$ is rational, that is, there exist integers a, b, $b \neq 0$, reduced, such that $\frac{p}{q} = \sqrt{6}$, so $\left(\frac{p}{q}\right)^2 = 6$. Then

$p^2 = 6q^2$ so $6 \mid p^2$. By part a), this implies that $6 \mid p$, so $p = 6k$ for some $k \in \mathbb{Z}$. Then

$(6k)^2 = 6q^2$, $36k^2 = 6q^2$, so $q^2 = 6k^2$ and $6 \mid q^2$

But this means that $6 \mid p$. We get that p and q are both divisible by 6, contradicting assumption that $\frac{p}{q}$ is reduced.

6. (6pts) Use the triangle inequality to prove that for all real numbers c, d,

$|c+1-(d+3)| \leqslant |c-d|+2$. Triangle inequality: $|a+b| \leq |a| + |b|$

$$|c+1-(d+3)| = |c-d+1-3| = |c-d-2| = |(c-d)+(-2)| \leq |c-d| + |-2|$$
$$= |c-d| + 2$$