Mathematical Reasoning — Exam 1
MAT 312, Fall 2015 — D. Ivanšić

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Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If the square of an integer is greater than one, then the integer is greater than one.

False: $(-2)^2 = 4 > 1$
but $-1 < 1$

2. (2pts) The Earth is round or the moon is closer than 1,000 miles away.

True: at least one of the statements (Earth is round) is true in an OR statement

3. (3pts) (universal set=$\mathbf{R}$) $3x - 1 = 3(x+2) - 7$ – open sentence

$\{x \mid 3x-1 = 3(x+2)-7\} = \mathbb{R}$

$3x-1 = 3x-1$
true for all x

4. (3pts) (universal set=$\mathbf{Z}$) $|x| \leq 10$ and x is divisible by 4. – open sentence

$\{x \mid |x| \leq 10 \text{ and } x \text{ is div. by } 4\}$
$= \{-8, -4, 0, 4, 8\}$

5. (4pts) For every $x \in \mathbf{R}$, $x^2 + 6x + 11 > 0$.

True: $x^2+6x+11 = x^2+6x+9-9+11 = \underbrace{(x+3)^2}_{\geq 0} + \underset{>0}{2} > 0$

Negate the following statements.

6. (3pts) President Xi supports building a Disney resort in China or puts an order for 300 Boeing airplanes.

President Xi doesn't support building a Disney Resort in China, or he doesn't put an order of 300 Boeing airplanes

7. (3pts) If the Pope says to look after the poor, then I will volunteer for a charity.

The Pope says to look after the poor and I do not volunteer for a charity.

8. (6pts) Use a truth table to prove that $(P \Longrightarrow Q) \vee (Q \Longrightarrow P)$ is a tautology. (Use however many columns you need.)

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \vee (Q \Rightarrow P)$					
T	T	T	T	T					
T	F	F	T	T					
F	T	T	F	T					
F	F	T	T	T					

9. (12pts) Use previously proven logical equivalences to prove the equivalence $P \vee (Q \Longrightarrow R) \equiv (Q \Longrightarrow P) \vee R$. Do not use a truth table.

$$P \vee (Q \Rightarrow R) \equiv P \vee (\neg Q \vee R)$$
$$\equiv (P \vee \neg Q) \vee R$$
$$\equiv (\neg Q \vee P) \vee R$$
$$\equiv (Q \Rightarrow P) \vee R$$

10. (4pts) Write the converse and contrapositive of the statement: if an integer is divisible by 6, then it is divisible by 3.

Converse: If an integer is divisible by 3, then it is divisible by 6

Contrapositive: If an integer is not divisible by 3, then it is not divisible by 6.

11. (10pts) Suppose the following statements are true:
If Benny gets a ball, then he doesn't get a bicycle.
If Benny gets a bicycle, then he gets a helmet.
Benny didn't get a helmet and he didn't get a ball.

		True statements
Benny gets ball	P	$P \Rightarrow \neg Q$
Benny gets bicycle	Q	$Q \Rightarrow R$
Benny gets helmet	R	$\neg P \wedge \neg R$

Determine truth value of the following statements and justify.

Benny got a ball.

False

Since $\neg P \wedge \neg R$ is true, but $\neg P$ and $\neg R$ are true.
$\neg P$ = Benny didn't get a ball

Benny got a bicycle.

False

$\neg R$ is true by argument at left.
$\neg R \Rightarrow \neg Q$ is true as the contrapositive of $Q \Rightarrow R$. Since $\neg R$ is true, so is $\neg Q$ = Benny didn't get a bicycle

12. (4pts) Use set builder notation to write the set $\{-1, 8, -27, 64, -125, \dots\}$.

cubes with alternating signs

$\{(-1)^m m^3 \mid m \in \mathbb{N}\} = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} \text{ s.t. } n = (-1)^k k^3\}$

13. (7pts) A function $f : (a, b) \to \mathbf{R}$ is said to be differentiable on (a, b) if $f'(c)$ exists for every c in (a, b).
a) Write the definition using symbols.
b) Negate the definition using symbols.
c) Finish the sentence: "A function f is not differentiable on (a, b) if ..."

a) f is differentiable on (a,b) if $\forall c \in (a,b)$, $f'(c)$ exists.

b) f is not differentiable on (a,b) if $\exists c \in (a,b)$ s.t. $f'(c)$ doesn't exist.

c) f is not differentiable on (a,b) if there exist a number $c \in (a,b)$ s.t. $f'(c)$ doesn't exist.

14. (10pts) For every natural number m, there exists a natural number n such that mn is a square of a natural number.
a) Write this statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

a) $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})$ (mn is a square of a natural number)

b) $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})$ (mn is not a square of a natural number)

c) There exists a natural number m s.t. for every natural number n, mn is not a square of a natural number.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(x + y^2 \geq 7)$$

a) If $x = 3$, is the statement true? $x = 3$, $(\forall y \in \mathbb{R})(3 + y^2 \geq 7)$ False, $y = 1$, $3 + 1^2 < 7$ (counter example)
b) If $x = 8$, is the statement true? $x = 8$, $(\forall y \in \mathbb{R})(8 + y^2 \geq 7)$ True, since $8 > 7$ and $y^2 \geq 0$, so $8 + y^2 \geq 7$
c) Find the truth set (the x's) of the above statement.

c) Need x's so that $x + y^2 \geq 7$ for all y's

$y^2 \geq 7 - x$ for all y's

Since we can always put in $y = 0$, we must have $0 \geq 7 - x$, so $x \geq 7$.
Conversely, if $x \geq 7$, then $x + y^2 \geq 7$, since $y^2 \geq 0$
Truth set $= \{x \in \mathbb{R} \mid x \geq 7\} = [7, \infty)$

16. (15pts) We will call an integer n type-0, type-1, type-2,..., or type-9 if it can be written in the form $n = 10k$, $n = 10k+1$, $n = 10k+2,\ldots$, or $n = 10k+9$, respectively, for some integer k. Use this idea to show the following: if an integer m ends with 4, and an integer n ends with 7, then the integer $m^2 + mn + n$ ends with 1. Start with a know-show table if you find it helpful.

Note: an integer ends with r if it is of type r.

Suppose m ends with 4 and n ends with 7. Then m is of type 4 and n is of type 7, so there exist integers k, l s.t. $m = 10k+4$ and $n = 10l+7$.

Now, $m^2 + mn + n = (10k+4)^2 + (10k+4)(10l+7) + 10l+7$

$= 100k^2 + 80k + 16 + 100kl + 40l + 70k + 28 + 10l + 7$

$= 100k^2 + 150k + 100kl + 50l + 51$

$= 100k^2 + 150k + 100kl + 50l + 50 + 1$

$= 10(10k^2 + 15k + 10kl + 5l + 5) + 1$

Since the number in parentheses is an integer, we have written $m^2 + mn + n$ in form $10q + 1$, q an integer, so it is of type 1, that is, it ends with a 1.

Bonus. (10pts) Show by example that quantifiers are not "commutative." That is, give an example of an open sentence $P(x, y)$ about real numbers x and y, so that the statements below have opposite truth values (justify why they do).

$(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})P(x, y)$ Let $P(x,y) \equiv x + y = 5$ $(\exists y \in \mathbf{R})(\forall x \in \mathbf{R})P(x, y)$

$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 5)$

This is true, since for every x the equation $x + y = 5$ has a solution in y, it is $y = 5 - x$

$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = 5)$

This is false. Take any y. Then for $x = 5 - y$, $x + y = 5$, but for $x = 4 - y$, $x + y = 4$, so $x + y = 5$ is not true for all x.