Topics in Topology — Exam 1
MAT 716, Spring 2014 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define the product topology on the product of topological spaces $\prod_{\alpha\in\Lambda} X_\alpha$.

Theory 2. (3pts) Define a convergent sequence in a topological space.

Theory 3. (3pts) State the theorem that tells us when a product space $X \times Y$ is metrizable.

Type A problems (5pts each)

A1. Show that $(3, \infty)$ is not a compact subspace of $(\mathbf{R}, \mathcal{C})$.

A2. Let $f : [1, 4] \times [3, 5] \to \mathbf{R}$ be a continuous function. Show that f achieves both its minimum and maximum value.

A3. Is $\prod_{n\in\mathbf{N}} W_n$, where $W_n = [-1, 1] - \{(-1)^n\}$ a connected space? W_n has the usual topology, and the product has the product topology.

A4. Show that if a finite topological space is metrizable, then it has the discrete topology.

A5. Consider the sequence of real numbers that form the digits of π: $3, 1, 4, 1, 5, 9, 2, \dots$ Show that this sequence is convergent in $(\mathbf{R}, \mathcal{C})$.

A6. Which of the following metric spaces (with the usual metric inherited from $\mathbf{R}$) is complete? Justify. a) $[3, 13] \cup [20, 21]$ b) $(-1, 1)$

Type B problems (8pts each)

B1. Is $[1, 4]$ a compact subset of $(\mathbf{R}, \mathcal{H})$? Justify.

B2. Show that the function $f : \prod_{n\in\mathbf{N}} \mathbf{R} \to \prod_{n\in\mathbf{N}} \mathbf{R}$ given by $f(x_1, x_2, x_3, x_4, \dots) = (x_2, x_1, x_4, x_3, \dots)$ is a homeomorphism.

B3. Let $F_\alpha \subseteq X_\alpha$ be a closed for every $\alpha \in \Lambda$. Show directly (i.e. do not use any statements from homework) that the set $\prod_{\alpha\in\Lambda} F_\alpha$ is closed in the product space $\prod_{\alpha\in\Lambda} X_\alpha$.

B4. Suppose metrics d and e are given on a set X that satisfy: there exist constants m, M such that $md(x, y) \le e(x, y) \le Md(x, y)$ for all $x, y \in X$. (Such metrics are called equivalent.) Show that d and e induce the same topology on X.

B5. Give an example of a sequence that converges in $(\mathbf{R},\mathcal{U})$, but does not converge in $(\mathbf{R},\mathcal{H})$. Is there a sequence that converges in $(\mathbf{R},\mathcal{H})$, but does not converge in $(\mathbf{R},\mathcal{U})$? Justify.

B6. Is $\mathbf{N} \subseteq \mathbf{R}$ with the usual metric inherited from $\mathbf{R}$ complete?

Type C problems (12pts each)

C1. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in $\prod_{\alpha\in\Lambda} X_\alpha$ (with product topology) and $x \in \prod_{\alpha\in\Lambda} X_\alpha$. Show that x_n converges to x if and only if $(x_n)_\alpha$ converges to x_α for every $\alpha \in \Lambda$.

C2. Let (X, d) be a metric space and $x_0, x_1 \in X$ be some fixed points.
a) Show that the function $f : X \to \mathbf{R}$, $f(x) = d(x, x_0)$ is continuous.
b) Use a) to easily show that the set $\{x \in X \mid d(x, x_0) = d(x, x_1)\}$ is closed.

Topics in Topology — Exam 2
MAT 716, Spring 2014 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define the quotient topology on a set X^*, if a topological space X and a surjective map $p : X \to X^*$ are given.

Theory 2. (3pts) Define a homotopy between maps $X \to Y$.

Theory 3. (3pts) Define β_h and state the proposition giving an explicit isomorphism between fundamental groups of X with different basepoints.

Type A problems (5pts each)

A1. Let X be a hexagon in the plane with the relative topology from $\mathbf{R}^2$ (interior of the hexagon is included). Identify opposite sides of the hexagon with identification (arrows) occuring in same direction. What is the resulting space?

A2. Let X be a compact space and $p : X \to X^*$ a surjective map. If X^* is given the quotient topology, show that X^* is compact.

A3. Let X^* be the quotient space resulting from identifying $X = [0,1] \times [0,1]$ using the relations $(0,t) \sim (1-t,1)$, $(s,0) \sim (1,1-s)$. Draw the picture of identification and conclude what X^* is.

A4. The Möbius strip deformation retracts to its center circle. Illustrate this deformation retraction by tracing the path of some of the points.

A5. $X \subseteq \mathbf{R}^n$ is star-shaped if there exists an $x_0 \in X$ such that for every $x \in X$, the line-segment from x_0 to x is contained in X. Write a formula for the deformation retraction of X to x_0.

A6. Let $f : X \to Y$ be a homotopy equivalence and $g, h : Y \to Z$ two maps. Show: if $gf \simeq hf$, then $g \simeq h$, i.e. you may "cancel" a homotopy equivalence.

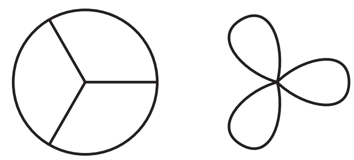
Type B problems (8pts each)

B1. Let C be the Cantor set, $D = \{0,2\}$, $f : \prod_{n\in\mathbf{N}} D \to C$ the known homeomorphism $f(a_1, a_2, \dots) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. The set $U = \left[\frac{2}{9}, \frac{3}{9}\right] \cap C$ is open in C because it is the intersection of a slightly larger open interval with C. What is the open set $f^{-1}(A)$?

B2. Let X be the rectangle $[-1,1] \times [0,1]$ with the relative topology from $\mathbf{R}^2$. Define an equivalence relation: $x \sim y$ if $x_1 = y_1 = 0$. Identify the familiar quotient space given by this relation and then show precisely that the quotient topology on this space is the same as the usual one.

B3. Show that a genus-2 surface with 3 crosscaps is homeomorphic to the sphere with 7 crosscaps. (Don't just quote the problem from homework - show it effectively).

B4. Define the homotopy equivalence between the spaces in the picture and justify with pictures why it is a homotopy equivalence. (Interiors are not included.)



B5. Let $f_0, f_1 : X \to Y$ be homotopic maps and $g : W \to X$. Show by explicitly constructing a homotopy that $f_0 g$ and $f_1 g$ are homotopic.

B6. Recall that $\mathbf{R}\mathrm{P}^2$ is S^2 with identification $x \sim -x$, let $p : S^2 \to \mathbf{R}\mathrm{P}^2$ be the surjection that induces the quotient topology. Let path f be half of the equatorial circle $S^1 \subseteq S^2$. Show that $g = pf$ is a loop in $\mathbf{R}\mathrm{P}^2$ that is homotopic to the loop $\overline{g}$. Give the explicit homotopy. Use this to conclude that $[g]^2 = 1$ in $\pi_1 \mathbf{R}\mathrm{P}^2$.

Type C problems (12pts each)

C1. Let $X = \left(\bigcup_{n \in \mathbf{N}} \left\{ \frac{1}{n} \right\} \times [0,1] \right) \bigcup (\{0\} \times [0,1]) \bigcup ([0,1] \times \{0\})$ with the relative topology from $\mathbf{R}^2$. Show that X deformation retracts to any point in $[0,1] \times \{0\}$, but not to any point in $\{0\} \times (0,1]$

Topics in Topology — Exam 3, take-home MAT 716, Spring 2014 — D. Ivanšić

Name: ______________________

Show all your work!

The rules: work on your own and do not discuss these problems with your classmates or anyone else. If necessary, come to me for hints. Do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted. You may use Hatcher's or Baker's books and lecture notes. On solutions, you may quote theorems from those, but you may not quote homework problems as all or part of a solution.

A1. If X is the Möbius strip, show that $\pi_1 X$ is isomorphic to $\mathbf{Z}$.

A2. It is an algebraic fact that any homomorphism $\mathbf{Z} \to \mathbf{Z}$ has the form $z \mapsto d \cdot z$ for some constant d. Show that every homomorphism $\pi_1 S^1 \to \pi_1 S^1$ can be realized as the induced map ϕ_* of a map $\phi : S^1 \to S^1$.

A3. Show that there is no retraction from the solid torus $S^1 \times D^2$ to its boundary $S^1 \times S^1$.

A4. Let a and b be two points in $\mathbf{R}^2$. Show that $\pi_1(\mathbf{R}^2 - \{a, b\})$ is not trivial.

Type B problems (8pts each)

B1. Let $A \subseteq X$, $x_0 \in A$. If $r : X \to A$ is a retraction, show that r_* is surjective. Furthermore, if X deformation retracts to A via the continuous family f_t where $f_0 = 1_X$ and $f_1(X) \subseteq A$ show that $f_{1*} : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is an isomorphism.

B2. Suppose loop f_1, based at x_0, is homotopic to the constant loop f_0 via a homotopy f_t that doesn't keep the basepoint fixed, i.e. $f_t(0) = f_t(1)$ for all t, but $f_t(0)$ need not be constant. Show that f_1 is homotopic to a constant loop via a homotopy g_t that keeps the basepoint fixed (i.e. $g_t(0) = g_t(1) = x_0$ for all t).

B3. Let $A \neq S^1$ be any closed arc on the circle. Show that S^1 retracts to A but does not deformation retract to A.

B4. Let $s_0 \in S^1$ and let S be the boundary circle of D^2. Show that there is no retraction $S^1 \times D^2 \to \{s_0\} \times S$, i.e. from the solid torus to a circle on the torus.

Type C problems (12pts each)

C1. For this problem, regard $\mathbf{R}\mathrm{P}^2$ as a sphere with a crosscap, that is, union of a Möbius strip X with a D^2 with their boundary circles identified. Show that $\pi_1 \mathbf{R}\mathrm{P}^2$ has no more than two elements by using the following steps (this comes out easily from van Kampen's theorem in 1.2, but its use is not allowed in this problem):

a) Show that $i : X \to \mathbf{R}\mathrm{P}^2$ induces a surjective map $i_* : \pi_1 X \to \pi_1 \mathbf{R}\mathrm{P}^2$. (Use the fact from the proof of $\pi_1 S^n = 1$: any loop in $\mathbf{R}\mathrm{P}^2$ may be homotoped off a point, in particular, off $0 \in D^2$).
b) If f is the center circle of X, show that $f \cdot f$ is homotopic to a constant map in $\mathbf{R}\mathrm{P}^2$.
c) Use i_* to reach the conclusion.

C2. Show that $\pi_1(\mathbf{R}^2 - \mathbf{Q} \times \{0\})$ is uncountable by constructing an injective map from an uncountable set into this group. Hint: $\pi_1(\mathbf{R}^2$ – one point) is used.

Topics in Topology — Final Exam **MAT 716, Spring 2014 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Describe the construction of the Cantor set as a subset of $[0, 1]$.

Theory 2. (3pts) Define a homotopy between paths f_0, $f_1 : I \to X$.

Theory 3. (3pts) Define β_h and state the proposition giving an explicit isomorphism between fundamental groups of X with different basepoints.

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A1. Let X be a hexagon in the plane with the relative topology from $\mathbf{R}^2$ (interior of the hexagon is included). Identify opposite sides of the hexagon with identification (arrows) occuring in same direction. What is the resulting space?

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A6. Let a and b be two points in S^2. Find $\pi_1(S^2 - \{a, b\})$.

Type B problems (8pts each)

B1. Let X be the rectangle $[-1, 1] \times [0, 1]$ with the relative topology from $\mathbf{R}^2$. Define an equivalence relation: $x \sim y$ if $x_1 = y_1 = 0$. Identify the familiar quotient space given by this relation and then show precisely that the quotient topology on this space is the same as the usual one.

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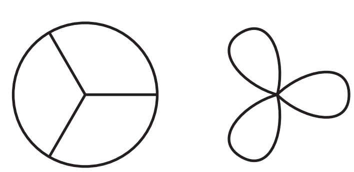

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B7. Let $X \subseteq \mathbf{R}^3$ be $X = S^2 \cup \{\text{the diameter joining } (0,0,-1) \text{ and } (0,0,1)\}$, and let A be the equatorial circle of the sphere lying in the x_2x_3-plane. Show that there is no retraction from X to A.

B8. Recall that $\mathbf{R}\mathrm{P}^2$ is S^2 with identification $x \sim -x$, let $p : S^2 \to \mathbf{R}\mathrm{P}^2$ be the surjection that induces the quotient topology. Let path f be half of the equatorial circle $S^1 \subseteq S^2$. Show that $g = pf$ is a loop in $\mathbf{R}\mathrm{P}^2$ that is homotopic to the loop $\overline{g}$. Give the explicit homotopy. Use this to conclude that $[g]^2 = 1$ in $\pi_1 \mathbf{R}\mathrm{P}^2$.

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