Topics in Topology — Problem Farm MAT 716, Spring 2014 — D. Ivanšić

Cantor Set

Note: A1 and B1 are general statements not connected to the Cantor set.

Type A problems (5pts each)

A1. (The pasting lemma) Let X be a topological space and $X = \bigcup_{i=1}^{n} F_i$, where F_i is a closed subset of X, $i = 1, \dots, n$. Let $f_i : F_i \to Y$ be continuous functions (the topology on F_i is the relative topology) such that for every $k, j = 1, \dots, n$ if $x \in F_j \cap F_k$, then $f_j(x) = f_k(x)$. This allows us to define the function $f : X \to Y$ by setting $f(x) = f_i(x)$, if $x \in F_i$.
a) Show that f is continuous.
b) Formulate a similar lemma for open subsets (you don't have to prove it). Can we have infinitely many subsets in this case?

Type B problems (8pts each)

B1. Let X_α, $\alpha \in \Lambda$ be a collection of mutually disjoint topological spaces. In such a case the union is written as $\bigsqcup_{\alpha \in \Lambda} X_\alpha$ ("disjoint union"). If X is this union, let $\mathcal{T} = \{U \subseteq X \mid U \cap X_\alpha \text{ is open in } X_\alpha\}$.
a) Show $\mathcal{T}$ is a topology on X.
b) Show $f : X \to Y$ is continuous if and only if $f|_{X_\alpha}$ is continuous for every $\alpha \in \Lambda$.
c) Show that for every $\alpha \in \Lambda$, X_α is both open and closed in X.
Note: one can form the disjoint union of identical spaces, for example $\bigsqcup_{n \in \mathbf{N}} \mathbf{R}$, by imagining we took disjoint homeomorphic copies of the same space. The formal way to do this is to take $\mathbf{N} \times \mathbf{R}$, and set $\bigsqcup_{n \in \mathbf{N}} \mathbf{R} = \mathbf{N} \times \mathbf{R}$. Clearly $\mathbf{N} \times \mathbf{R}$ can be viewed as a disjoint union of countably many copies of $\mathbf{R}$. In the more general case where sets in the collection are not all same, let $Z = \bigcup_{\alpha \in \Lambda} X_\alpha$, and define X as the following subset of $\Lambda \times Z$: $X = \bigcup_{\alpha \in \Lambda} \{\alpha\} \times X_\alpha$. Then X is a union of disjoint copies of $X_\alpha = \{\alpha\} \times X_\alpha$ and we proceed as above.

B2. Let C be the Cantor set, $f : \{\text{sequences of 0'2 and 2's}\} \to C$ the map

$$f(a_1, a_2, \dots) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Let C_n be the union of 2^n intervals as in lectures. Show that $f(a_1, a_2, \dots)$ is an endpoint of an interval in C_n if and only if the sequence $a_1, a_2, \dots$ is all zeroes or all twos starting with some index (i.e. except for finitely many terms, the sequence is all zeroes or all twos)

B3. Let D be the two-point set $\{0, 2\}$ with the discrete topology. Show that the Cantor set is homeomorphic to $\prod_{n \in \mathbf{N}} D$. (Hint: use the map f from **B1** and show it is continuous in the formulation of Theorem 3.2.9e).

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Quotient Topology

Type A problems (5pts each)

A1. If $p : X \to X^*$ is surjective and X is a topological space, show that the collection $\mathcal{T} = \{V \subseteq X^* \mid p^{-1}(V) \text{ is open in } X\}$ is a topology (that's the quotient topology).

A2. If $p : X \to X^*$ is surjective, show that the quotient topology on X^* is the finest topology for which p is continuous. That is, if $\mathcal{T}$ is any topology on X^* in which p is continuous, then $\mathcal{T}$ is contained in the quotient topology.

A3. Give an example of a Hausdorff space X and a surjective map $p : X \to X^*$ so that the quotient space X^* is not Hausdorff.

A4. Let X be a topological space, $p : X \to Y$ and $q : Y \to Z$ surjections. Give Y the quotient topology induced by p. Then Z can be given the quotient topologies induced by q or by $q \circ p$. Show that these topologies are the same.

A5. Consider the function $p : \mathbf{R} \to \{a, b, c\}$ given by:

$$f(x) = \begin{cases} a, & \text{if } x < -3 \\ b, & \text{if } -3 \leq x \leq 3. \\ c, & \text{if } x > 3. \end{cases}$$

List the open sets in the quotient topology on $\{a, b, c\}$.

A6. Let X be a hexagon in the plane with the relative topology from $\mathbf{R}^2$ (interior of the hexagon is included). Identify opposite sides of the hexagon with identification (arrows) occuring in same direction. What is the resulting space?

Type B problems (8pts each)

B1. Let a surjective $p : X \to X^*$ induce the quotient topology on X^*. Give an example where p is not an open map and one where p is not a closed map (open, closed maps send open sets to open sets or closed sets to closed sets, respectively).

B2. Consider the projection map from a product of topological spaces $p_\beta : \prod_{\alpha \in \Lambda} X_\alpha \to X_\beta$. Show that the quotient topology on X_β induced by p_β is the same as the original topology on X_β. (Helpful fact: p_β is an open map.)

B3. Let D^2 be the unit disk $\{(x_1, x_2) \in \mathbf{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ with the relative topology from $\mathbf{R}^2$. Define an equivalence relation: $x \sim y$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Identify the familiar quotient space given by this relation and then show precisely that the quotient topology on this space is the same as the usual one.

B4. Let D^2 be the unit disk with the relative topology from $\mathbf{R}^2$. Define an equivalence relation: $x \sim y$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$. Identify the familiar quotient space given by this relation and then show precisely that the quotient topology on this space is the same as the usual one.

B5. Consider all the possible ways the sides of a rectangle can be identified in pairs (like for the torus or the Klein bottle, but keep in mind that adjacent sides may be identified). What spaces result from these idenfications?

B6. Show by induction on g that a genus-g surface with k crosscaps is homeomorphic to the sphere with $k + 2g$ crosscaps.

Type C problems (12pts each)

C1. Let $p : X \to X^*$ be surjective, and give X^* the quotient topology.
a) Suppose $B \subseteq X^*$ is a closed or an open set. B may be given the relative topology or the quotient topology induced by the function $p|_{f^{-1}(B)} : f^{-1}(B) \to B$. Show these topologies are the same.
b) If $A \subseteq X$ is a closed or an open set with the property that $p^{-1}(p(A)) = A$, then $p(A)$ may be given the relative topology or the quotient topology induced by the function $p|_A$. Show these topologies are the same.
c) Give an example of a closed or an open set A for which the relative topology on $p(A)$ is not the same as the quotient topology induced by $p|_A$. (Hint: consider some stock examples and an A where $p : A \to X^*$ is surjective, but $A \neq X$.)

C2. Let X be a $2k$-gon in the plane with the relative topology from $\mathbf{R}^2$ (interior of the polygon is included). Identify opposite sides of X with identification (arrows) occuring in same direction. Use induction to find the resulting space. (Hint: examine cases $k = 3, 4, 5, 6$ to get an idea.)

C3. Let X be a $2k$-gon in the plane with the relative topology from $\mathbf{R}^2$ (interior of the polygon is included). Identify $0 \leq j < k$ pairs of opposite sides of X with identification (arrows) occuring in the same direction, and $k - j > 0$ pairs of opposite sides with identification (arrows) occuring in the opposite direction. Show that the resulting space is a sphere with $j + 1$ crosscaps by following these steps:

a) Show you can reduce the problem either to

(i) a polygon (with fewer sides) where no two adjacent sides are identified to opposing sides in the opposite direction (adjacent sides may still pair to opposing sides in the same direction), or

(ii) (when $j = 0$) a 2-gon with the two sides paired in the opposite direction, in which case the resulting space is obvious.

b) In case (i), cut the polygon and rearrange it to see that you get the sphere with $j + 1$ crosscaps.

C4. Generalize the previous two problems to a $2k$-gon where pairs of sides, not necessarily opposing ones, are identified in one of the two possible ways.