Advanced Calculus 1 — Exam 1
MAT 525/625, Fall 2014 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the ϵ-neighborhood of a.

Theory 2. (3pts) Define the supremum of a set.

Theory 3. (3pts) If S and T are sets and $T \subseteq S$, state the theorem (2 claims) that relates countability of S to countability of T.

Type A problems (5pts each)

A1. Show using Mathematical Induction that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbf{N}$.

A2. Let A be countable and B uncountable. Prove that $A \cup B$ is uncountable and $A \cap B$ is countable.

A3. If $a \in \mathbf{R}$, use algebraic properties of $\mathbf{R}$ to show that $-(-a) = a$, and $1/(-a) = -(1/a)$ (assuming $a \neq 0$).

A4. Show that $x + \frac{1}{x} \geq 2$ for all $x > 0$ and that equality holds if and only if $x = 1$.

A5. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = (-1, 4]$ b) $S = \left\{ \frac{1}{\sqrt{n}} \mid n \in \mathbf{N} \right\}$

Type B problems (8pts each)

B1. Let A be a denumerable collection of lines in the plane. Show that the set of all points that are intersections of any two of those lines is countable.

B2. Let $a \in \mathbf{N}$ be an even number that is not divisible by 4. Show that $\sqrt{a}$ is not a rational number.

B3. Determine and sketch the set of points in the plane satisfying $2|x| + |y| \leq 3$.

B4. Let $S = (1, \infty)$. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

B5. Let S be a nonempty set that is bounded above. Show that $\sup e^S = e^{\sup S}$. (As expected, $e^S = \{e^s \mid s \in S\}$.)

Type C problems (12pts each)

C1. Let $S = \{x \in \mathbf{R} \mid x^2 + bx + c = 0 \text{ for some } b, c \in \mathbf{Z}\}$. That is, S is the set of all solutions of quadratic equations $x^2 + bx + c = 0$ with coefficients $b, c \in \mathbf{Z}$. Show that S is countable.

C2. Show that the set of all functions $\mathbf{N} \to \{0, 1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0, 1\}$ *may be thought of as a sequence of 0's and 1's.*

C3. Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function, and S a nonempty set that is bounded above. Although $\sup f(S) = f(\sup S)$ is true for reasonable functions f, find a function f — obviously, unreasonable — for which this is false.

Advanced Calculus 1 — Exam 1 make-up **MAT 525/625, Fall 2014 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a set S is denumerable.

Theory 2. (3pts) Define when a set is bounded above and when it is bounded below.

Theory 3. (3pts) State the theorem on the density of $\mathbf{Q}$.

Type A problems (5pts each)

A1. Show using Mathematical Induction that $5^n + 12n - 1$ is divisible by 16 for all $n \in \mathbf{N}$.

A2. Let S be the collection of all lines in $\mathbf{R}^2$ whose x- and y-intercepts are natural numbers. Show that S is countable.

A3. If $a, b \in \mathbf{R}$, use algebraic properties of $\mathbf{R}$ to show that $-(a + b) = (-a) + (-b)$, and $1/(ab) = (1/a)(1/b)$ (assuming $a, b \neq 0$).

A4. Show that $x + \frac{1}{x} \geq 2$ for all $x > 0$ and that equality holds if and only if $x = 1$.

A5. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = (-3, \infty)$ b) $S = \left\{1 + \frac{1}{n} \mid n \in \mathbf{N}\right\}$

Type B problems (8pts each)

B1. Let A be countable and B uncountable. Prove that $A \setminus B$ is countable and $B \setminus A$ is uncountable.

B2. Let $a \in \mathbf{N}$ be an even number that is not divisible by 4. Show that $\sqrt{a}$ is not a rational number.

B3. Determine and sketch the set of points in the plane satisfying $|y| - 2|x| \leq 3$.

B4. Let $S = \left\{m + \frac{1}{n} \mid m, n \in \mathbf{N}\right\}$. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

B5. Let S be a nonempty set that is bounded above. Show that $\sup S^3 = (\sup S)^3$. (As expected, $S^3 = \{s^3 \mid s \in S\}$.)

Type C problems (12pts each)

C1. Let $S = \{x \in \mathbf{R} \mid x^2 + bx + c = 0 \text{ for some } b, c \in \mathbf{Z}\}$. That is, S is the set of all solutions of quadratic equations $x^2 + bx + c = 0$ with coefficients $b, c \in \mathbf{Z}$. Show that S is countable.

C2. Show that the set of all functions $\mathbf{N} \to \{0, 1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0, 1\}$ *may be thought of as a sequence of 0's and 1's.*

C3. Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function, and S a nonempty set that is bounded above. Although $\sup f(S) = f(\sup S)$ is true for reasonable functions f, find a function f — obviously, unreasonable — for which this is false.

Advanced Calculus 1 — Exam 2
MAT 525/625, Fall 2014 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a subsequence of a sequence.

Theory 2. (3pts) Define when a sequence tends to ∞.

Theory 3. (3pts) State the Monotone Convergence Theorem.

Type A problems (5pts each)

A1. Use the definition of the limit to show $\lim \frac{2n+3}{n-7} = 2$.

A2. Find $\lim \sqrt[2n]{7n^3}$.

A3. If $0 < a < b$, find $\lim (a^n + b^n)^{\frac{1}{n}}$.

A4. Establish convergence or divergence of $x_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}$.

A5. If $\lim x_n = \infty$ and $\lim y_n = \infty$, prove using the definition that $\lim(x_n + y_n) = \infty$.

A6. Show $\lim \left(1 + \frac{1}{n}\right)^{n^2} = \infty$. Make sure you use only arguments we established.

Type B problems (8pts each)

B1. Use the squeeze theorem to determine $\lim \sqrt[n]{n^2 - 3n + 1}$.

B2. Let the sequence x_n be recursively given by: $x_1 = 1$, $x_{n+1} = \sqrt{7 + 2x_n}$. Show that this sequence converges and find its limit.

B3. Find $\lim \left(1 + \frac{1}{n^2}\right)^n$ and justify your reasoning. While this problem may be done fairly easily with "calculus 1" techniques, make sure you use only arguments we established (ask if unsure).

B4. Show that if a sequence (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim \frac{1}{x_{n_k}} = 0$.

B5. Use the definition to show that the sequence $x_n = \frac{n^2}{n^2+1}$ is Cauchy.

B6. Let the sequence x_n be recursively given by: $|x_1| \leq 1$, $x_{n+1} = \frac{1}{5}(x_n^3 + x_n - 1)$. Show that this sequence is contractive and write the equation that its limit satisfies (do not solve the equation, since it is not easy).

Type C problems (12pts each)

C1. Let the sequence x_n be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$.
a) Show for any $a > 0$: $\frac{1}{2}\left(a + \frac{5}{a}\right) \geq \sqrt{5}$.
b) Show that x_n is a contractive sequence (part a) will help).
c) Find $\lim x_n$ if it exists.

C2. Inspired by a student question, this problem is a negative version of the homework problem: if the even terms and the odd terms of a sequence converge to the same limit, then the sequence converges to this limit. We give an example of a sequence (x_n) that can be broken up into infinitely many "disjoint" subsequences, all of which converge to the same limit, but (x_n) itself diverges.

For every prime p build the set $A_p \subseteq \mathbf{N}$ inductively as follows:

$A_2 = \{2, 4, 6, \dots\}$, all multiples of 2.
$A_3 = \{3, 6, 9, \dots\} \setminus A_2$,
$A_5 = \{5, 10, 15, \dots\} \setminus (A_2 \cup A_3)$, and so on: if p is the first prime after q, then
$A_p = \{p, 2p, 3p, \dots\} \setminus (A_2 \cup A_3 \cup \dots \cup A_q)$.

a) Show that A_p is infinite.
b) Show that $A_p \cap A_q = \emptyset$ if $p \neq q$ and $\mathbf{N} = A_2 \cup A_3 \cup A_5 \cup \dots$.

Define the sequence: $x_n = \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not prime.} \end{cases}$

Now suppose the infinitely many elements of A_p are labeled in increasing order as $A_p = \{n_{p1}, n_{p2}, \dots, n_{pk}, \dots\}$. For every prime p, consider the subsequence X_p of (x_n) whose k-th term is $x_{n_{pk}}$ (we may write this as $X_{pk} = x_{n_{pk}}$).

c) Show that x_n diverges.
d) Show that all the sequences X_p converge to the same limit.

Advanced Calculus 1 — Exam 3
MAT 525/625, Fall 2014 — D. Ivanšić

Name: ____________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a cluster point of a set $A \subseteq \mathbf{R}$.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = \infty$ means.

Theory 3. (3pts) State sequential criterion for convergence for the case $\lim_{x\to c} f(x) = L$.

Type A problems (8pts each)

A1. Find the limits, if they exist (just a computation is expected): a) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$ b) $\lim_{x\to\infty} \dfrac{x^2-7x+12}{2x^2+5x+4}$.

A2. Does $\lim_{x\to 0} x\sin^2\left(\dfrac{1}{x}\right)$ exist? If yes, find it, if not, justify.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by: $f(x) = x$, if x is rational, $f(x) = x^2$ if x is irrational. find $\lim_{x\to 0} f(x)$ if it exists.

A4. Prove the limit theorem by definition: if $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to-\infty} g(x) = M$, then $\lim_{x\to-\infty} (f(x)+g(x)) = L+M$.

A5. Use the definition to show that $\lim_{x\to\infty} \dfrac{2x-4}{5x+2} = \dfrac{2}{5}$.

Type B problems (8pts each)

B1. Use the definition to show that $\lim_{x\to c}(x^3+2x-7) = c^3+2c-7$ for any $c \in \mathbf{R}$.

B2. Prove the extended limit law $L \cdot \infty = \infty$, where $L > 0$, that is, if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c}(f(x)g(x)) = \infty$.

B3. Suppose $\lim_{x\to 0+} xf(x) = L$, $L > 0$. Show that $\lim_{x\to 0+} f(x) = \infty$.

B4. Suppose $A \subseteq \mathbf{R}$ is a bounded set. If $\sup A \notin A$, show $\sup A$ is a cluster point of A. What can you say if $\sup A \in A$?

B5. Suppose $\lim_{x\to\infty} f(x) = \infty$ and that $g(x)$ is bounded on some interval (a, ∞). Show that $\lim_{x\to\infty} (f(x) - g(x)) = \infty$.

Type C problems (12pts each)

C1. Suppose $f, g : \mathbf{R} \to \mathbf{R}$, and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to L} g(x) = M$. Contrary to what one may think, $\lim_{x \to c} g(f(x)) = M$ is not always true. Give an example where it is not. (This one is subtle and hinges on the possibility that $g(L) \neq M$.)

Advanced Calculus 1 — Final Exam **MAT 525/625, Fall 2014 — D. Ivanšić**

Name: ______________________________

Show all your work!

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) Define the supremum of a set.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = L$ means.

Theory 3. (3pts) State the Bolzano-Weierstrass theorem for sequences.

Type A problems (5pts each)

A1. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = \mathbf{Q} \cap \left[\sqrt{5}, 7\right)$ b) $S = \left\{\frac{(-1)^n}{n} + \frac{(-1)^m}{m} \mid n, m \in \mathbf{N}\right\}$

A2. Use the definition of the limit to show $\lim \frac{2n-4}{5n+2} = \frac{2}{5}$.

A3. Establish convergence or divergence of $x_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}$.

A4. Find the limits, if they exist (just a computation is expected): a) $\lim_{x\to\infty} \dfrac{x-4}{\sqrt{x}-7}$ b) $\lim_{x\to 0+} \dfrac{\sqrt{5x+1}-1}{x^2}$.

A5. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by: $f(x) = x$, if x is rational, $f(x) = x^2$ if x is irrational. Find $\lim_{x\to 0} f(x)$ if it exists.

A6. Prove the extended limit law $\infty \cdot \infty = \infty$, where $L > 0$, that is, if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c} (f(x)g(x)) = \infty$.

Type B problems (8pts each)

B1. Let A be countable and B uncountable. Prove that $A \setminus B$ is countable and $B \setminus A$ is uncountable.

B2. Let S be a nonempty set that is bounded above. Show that $\sup S^3 = (\sup S)^3$. (As expected, $S^3 = \{s^3 \mid s \in S\}$.)

B3. Let the sequence x_n be recursively given by: $x_1 = 2$, $x_{n+1} = 5 - \frac{2}{x_n}$. Show that this sequence converges and find its limit.

B4. Use the definition to show that the sequence $x_n = \frac{n^2}{n^2+1}$ is Cauchy.

B5. Find $\lim \left(1 + \frac{1}{n^2}\right)^n$ and justify your reasoning. While this problem may be done fairly easily with "calculus 1" techniques, make sure you use only arguments we established (ask if unsure).

B6. Use the definition to show that $\lim_{x \to c}(x^3 + 3x^2 - 5x + 1) = c^3 + 3c^2 - 5c + 1$ for any $c \in \mathbf{R}$.

B7. Suppose $A \subseteq \mathbf{R}$ is a bounded set. If $\sup A \notin A$, show $\sup A$ is a cluster point of A. What can you say if $\sup A \in A$?

Type C problems (12pts each)

C1. Show that the set of all functions $\mathbf{N} \to \{0, 1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0, 1\}$ *may be thought of as a sequence of 0's and 1's.*

C2. Let the sequence x_n be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$.
a) Show for any $a > 0$: $\frac{1}{2}\left(a + \frac{5}{a}\right) \geq \sqrt{5}$.
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