Fall 2009: MAT 525, Exam 1

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the ϵ-neighborhood of a point $a \in \mathbf{R}$.

Theory 2. (3pts) Define when a set is bounded above and when it is bounded below.

Theory 3. (3pts) State the Completeness Property of $\mathbf{R}$.

Type A problems (5pts each)

A1. Let $f : A \to B$, and let G, $H \subseteq B$. Show that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

A2. If $a, b \in \mathbf{R}$, use algebraic properties of $\mathbf{R}$ to show that $(-1)a = -a$, and $1/(ab) = (1/a)(1/b)$ (assuming $a, b \neq 0$).

A3. Show that $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbf{R}$, and that equality holds if and only if $a = b$.

A4. Prove: if $a \in \mathbf{R}$ is such that $a < \epsilon$ for every $\epsilon > 0$, then $a \leq 0$.

A5. Find an upper and a lower bound of $S = (-2, 5]$. Justify if one or both do not exist. Then find $\inf S$ and $\sup S$ and justify.

Type B problems (8pts each)

B1. Show using Mathematical Induction that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbf{N}$.

B2. Let $a \geq 0$ and $b \geq 0$. Show that $a < b$ if and only if $a^2 < b^2$.

B3. Determine and sketch the set of points in the plane satisfying $|x| - |y| \geq 2$.

B4. Find an upper and a lower bound of $S = \{\sqrt{n} \mid n \in \mathbf{N}\}$. Justify if one or both do not exist. Then find $\inf S$ and $\sup S$ and justify.

B5. Let $S \subset (0, \infty)$ be a nonempty set that is bounded above. Show that $\inf \frac{1}{S} = \frac{1}{\sup S}$. (As expected, $\frac{1}{S} = \{\frac{1}{s} \mid s \in S\}$.)

Type C problems (12pts each)

C1. Show that for every $a \in \mathbf{R}$, $a \geq 0$, there exists a number so that $x^2 = a$.

C2. Suppose a set satisfies all the algebraic and order properties of $\mathbf{R}$. Show that it cannot satisfy the following "well-ordering" property: every nonempty set S bounded below has a least element. (Postulates existence of a minimal element, rather than an infimum.) *Hint: find a set S that violates the property. Justification needs to be based on properties of* $\mathbf{R}$ *rather than intuition.*

Fall 2009: MAT 525, Exam 2

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the limit of a sequence.

Theory 2. (3pts) Define a Cauchy sequence.

Theory 3. (3pts) State the Bolzano-Weierstrass theorem for sequences.

Type A problems (5pts each)

A6. If (x_n) and (y_n) are sequences such that $\lim x_n = 0$ and (y_n) is bounded, show that $\lim x_n y_n = 0$.

A7. Use the squeeze theorem to determine $\lim (n!)^{\frac{1}{n^2}}$.

A8. Find $\lim \left(1 + \frac{1}{3n}\right)^{9n}$ and justify your reasoning.

A9. Give an example of a sequence whose every convergent subsequence converges to 12, yet the sequence itself does not converge.

A10. Let (x_n) and (y_n) be sequences such that $\lim x_n = \infty$ and $x_n \leq y_n$ for all $n \in \mathbf{N}$. Use the definition to prove that $\lim y_n = \infty$.

Type B problems (8pts each)

B6. If (x_n) and (y_n) are convergent sequences, show that $\lim(x_n + y_n) = \lim x_n + \lim y_n$.

B7. Using the definition, prove the extended limit law $L \cdot \infty = \infty$, if $L > 0$. That is, if (x_n) and (y_n) are sequences such that $\lim x_n = L$, $L > 0$, and $\lim y_n = \infty$, then $\lim x_n y_n = \infty$.

B8. State and prove the Monotone Convergence Theorem.

B9. Let (x_n) be given inductively by $x_{n+1} = \sqrt{3 + 2x_n}$, with $x_1 > 0$. Show that (x_n) converges and find the limit.

B10. Let $x_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{n \cdot 2^n}$. Show that (x_n) is a Cauchy sequence.

B11. Let (x_n) be given inductively by $x_{n+1} = \frac{1}{7}(1 - x_n^2)$, with $|x_1| \leq 3$. Show that (x_n) converges and find the limit. *(Hint: contractive sequence.)*

Type C problems (12pts each)

C3. If (x_n) and (y_n) are convergent sequences, show that $\lim(x_n y_n) = \lim x_n \cdot \lim y_n$.

C4. Let $x_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n}$. Show that (x_n) is convergent.

C5. Determine $\lim \sqrt[n]{n!}$.

Fall 2009: MAT 525, Exam 3

Do all the theory problems. Then do at least five problems, one of which is of a different type than others (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define what $\lim_{x\to c} f(x) = L$ means.

Theory 2. (3pts) Define a cluster point of a subset $A \subset \mathbf{R}$.

Theory 3. (3pts) State the Intermediate Value Theorem.

Type A problems (5pts each)

A11. Find the limits, if they exist: a) $\lim_{x\to 5} \frac{\sqrt{5}-\sqrt{x}}{x-5}$ b) $\lim_{x\to 3} \frac{x^2-7x+12}{x-3}$.

A12. Use the definition to show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

A13. Does $\lim_{x\to 0} \cos^2 \frac{1}{x}$ exist? If yes, find it, if not, justify.

A14. Let $f, g : \mathbf{R} \to \mathbf{R}$ be functions such that f is continuous at c and g is continuous at $b = f(c)$. Show that $g \circ f$ is then continuous at c.

A15. Let $f : [a, b] \to \mathbf{R}$ be a function such that $f(x) < 0$ for all $x \in [a, b]$. Show there exists a number $w < 0$ such that $f(x) \le w$ for all $x \in [a, b]$. Give an example of a function $f : (a, b) \to \mathbf{R}$ for which the conclusion does not hold.

Type B problems (8pts each)

B12. Use the definition to show that $\lim_{x\to c}(x^2 + 4x + 7) = c^2 + 4c + 7$ for any $c \in \mathbf{R}$.

B13. Show that $\lim_{x\to\infty} (x^2 + x \sin x) = \infty$.

B14. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given by: $f(x) = x$, if x is rational, $f(x) = -x$ if x is irrational. Prove f is continuous at 0 and discontinuous at every other point.

B15. Let $f, g : A \to \mathbf{R}$ be functions such that $f(x) \le g(x)$ for all $x \in A$, and let c be cluster point of A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, use the definition to show that $L \le M$.

B16. Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Prove that f is bounded on $[a, b]$.

B17. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a number $c \in [0, 1]$ such that $f(c) = c$. *(Hint: consider the function $g(x) = f(x) - x$.)*

Type C problems (12pts each)

C6. Use the definition to show that $\lim_{x\to\infty} \frac{x^2 - x + 4}{2x^2 - 5x + 7} = \frac{1}{2}$.

C7. Let $f : (0, \infty) \to \mathbf{R}$, $f(x) = 0$ if x is irrational, $f(x) = \frac{1}{n}$ if x is a rational number represented by $\frac{m}{n}$ in reduced form. Show that f is continuous at every irrational point and discontinuous at every rational point.