Calculus 1 — Exam 1
MAT 250, Spring 2013 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim\limits_{x \to -2^-} f(x) =$

$\lim\limits_{x \to -2^+} f(x) =$

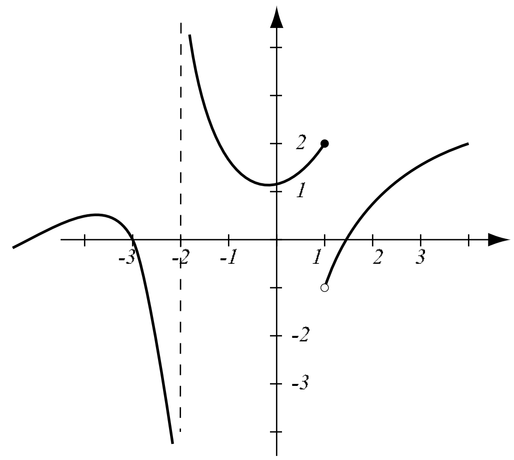
$\lim\limits_{x \to -2} f(x) =$

$\lim\limits_{x \to 1^+} f(x) =$

$\lim\limits_{x \to 1} f(x) =$

$f(1) =$

List points where f is not continuous and justify why it is not continuous at those points.



2. (8pts) Let $\lim\limits_{x \to 2} f(x) = 3$ and $\lim\limits_{x \to 2} g(x) = -1$. Use limit laws to find the limit below and show each step.

$$\lim_{x \to 2} \sqrt{\frac{xf(x) - 4}{x^3 + g(x)}} =$$

3. (10pts) Find $\lim\limits_{x \to 0} \dfrac{x^2}{4 + \sin\left(\frac{1}{x} + 3\right)}$. Use the theorem that rhymes with an exclamation conveying surprise and derision.

Find the following limits algebraically. Do not use the calculator.

4. (5pts) $\lim_{x \to 4} \dfrac{x^2 - 16}{x^2 - 5x + 4} =$

5. (7pts) $\lim_{x \to 13} \dfrac{\sqrt{x+3} - 4}{x - 13} =$

6. (6pts) $\lim_{x \to 0} \dfrac{\tan x}{x} =$

7. (7pts) $\lim_{x \to \infty} \dfrac{5x^2 - 3x + 1}{4x^3 - 4x^2 + 7} =$

8. (5pts) $\lim_{x \to 3^+} \dfrac{2x + 1}{3 - x} =$

9. (14pts) Use your calculator to find an interval of length at most 0.01 that contains the solution of the equation $x^3 - 4x^2 + 3x = 8$. Use the Intermediate Value Theorem to justify why your interval contains the solution.

10. (10pts) Consider the limit $\lim_{x\to 1} \dfrac{2^x - 2}{x - 1}$. Use your calculator to estimate this limit with accuracy 4 decimal points. Write a table of values that will justify your answer.

x	$\dfrac{2^x - 2}{x - 1}$

11. (12pts) Draw the graph of a function, defined on the interval $(-3,4)$ that exhibits the following features:

$\lim_{x\to\infty} f(x) = 3$

$\lim_{x\to-\infty} f(x) = 1$

$\lim_{x\to 0^-} f(x) = 4$

$\lim_{x\to 0^+} f(x) = -1$

$f(x)$ is left-continuous at $x = 0$

Bonus. (10pts) Show that $\frac{0}{0}$ is an indeterminate form. That is, come up with three pairs of functions $f(x)$, $g(x)$ such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ in every case, yet $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is different for the three cases. (Think of simple functions.)

Calculus 1 — Exam 2
MAT 250, Spring 2013 — D. Ivanšić

Name: ____________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left(4x^4 + \dfrac{4}{x^2} - x^2\sqrt{x} + \sqrt[4]{\pi}\right) =$

2. (5pts) $\dfrac{d}{du}\,(u^2 - 3u)\sqrt{4u-7} =$

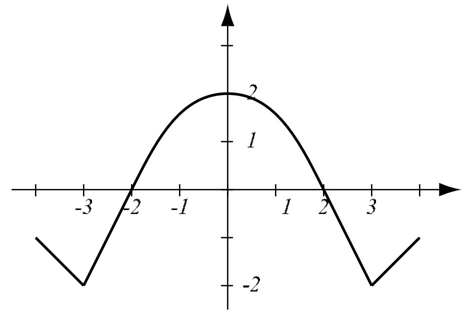
3. (6pts) $\dfrac{d}{dx}\,\dfrac{x-4}{x^2-3x+1} =$

4. (6pts) $\dfrac{d}{d\theta}\left(\cos^2\theta - \sin^2\theta\right) =$

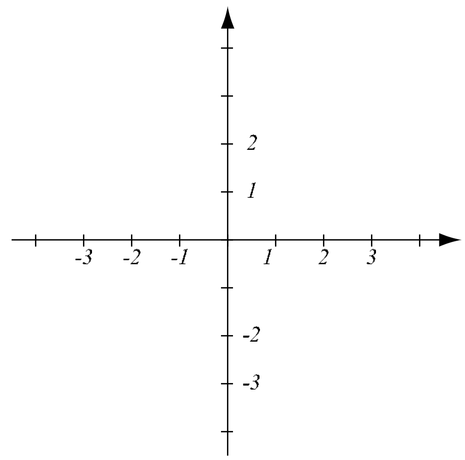
5. (7pts) $\dfrac{d}{dx}\sqrt{\tan\left(x^4+x^2+1\right)} =$

6. (8pts) Let $h(x) = f(x)\sin x$. Find the general expressions for $h'(x)$ and $h''(x)$ and simplify where appropriate.

7. (12pts) The graph of the function $f(x)$ is shown at right.



a) Where is $f(x)$ not differentiable?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?



8. (16pts) Let $f(x) = \dfrac{1}{3x+2}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(3, \frac{1}{11})$.

9. (16pts) A first-generation iPhone is thrown upwards from ground level with initial velocity 40m/s (what else to do with a first-generation device!). Its position is given by the formula $s(t) = -5t^2 + 40t$.

a) Write the formula for the velocity of the iPhone at time t.

b) When does the iPhone reach height 60 meters?

c) What is the velocity of the iPhone when it reaches height 60 meters on its way up? On its way down?

d) What is the height of the iPhone when its velocity is 10m/s?

10. (6pts) Consider the limit below. It represents a derivative $f'(a)$.

a) Find f and a.

b) Once you've found f and a, find the limit — it is equal to $f'(a)$!

$$\lim_{x \to \frac{\pi}{3}} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}}$$

11. (12pts) The temperature (in degrees Celsius) of water being heated in a pan is given by $f(t) = 10 + 30\sqrt{t}$, where t is in minutes.
a) What is the average rate of warming from $t = 4$ to $t = 6$? What are the units?
b) What is the instantaneous rate of warming when $t = 4$? What are the units?
c) Draw the graph of f and state the geometric interpretation of the numbers you got above.

Bonus. (10pts) Take the derivative of the function below and simplify. Go to town!

$$\frac{d}{dx}\sqrt{\frac{x+\sqrt{x}}{x-\sqrt{x}}} =$$

Calculus 1 — Exam 3 **MAT 250, Spring 2013 — D. Ivanšić**

Name: ____________________

Show all your work!

1. (14pts) Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$.

2. (12pts) Use implicit differentiation to find y'.

$\sin x + \cos y = \dfrac{x}{y}$

3. (6pts) Sketch the graphs of the following functions.

$y = a^x$, $a > 1$ $\qquad$ $y = a^x$, $0 < a < 1$ $\qquad$ $y = \log_a x$, $a > 1$

Find the following limits algebraically. The graphs you drew above may help you.

4. (2pts) $\lim\limits_{x \to \infty} \left(\frac{1}{3}\right)^x =$

5. (6pts) $\lim\limits_{x \to \infty} \dfrac{5^x + 4}{2 \cdot 5^x - 7} =$

6. (6pts) $\lim\limits_{x \to 0^+} e^{1 - \frac{1}{x}} =$

7. (14pts) One side of a right triangle is known to have length 6 feet. The other side has been measured to be 8 feet, with maximum error $\frac{1}{2}$ of an inch. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the length of the hypothenuse of this triangle.

8. (12pts) Estimate $\sqrt{4.5}$ using linear approximation by doing the following:
a) Write the linearization of the appropriate function at the appropriate point.
b) Use the linearization to estimate $\sqrt{4.5}$ and compare it to the calculator-given value 2.12132.

9. (12pts) Let $f(x) = x^2 - 10x + 20$, $x \leq 5$. Use the theorem on derivatives of inverses to find $(f^{-1})'(4)$.

10. (16pts) A cylindrical tank with flexible sides contains 45m^3 of water (the tank is not full). The radius of the tank is shrinking at the rate of 0.1 meters per minute. How fast is the water level rising when the radius is 3 meters? Recall the volume of a cylinder is given by $V =$ area of base $\times$ height.

Bonus. (10pts) Find the formula for the inverse of the function in problem 9. Use it to find $(f^{-1})'$ and $(f^{-1})'(4)$. You should get the same answer as in problem 9.

Calculus 1 — Exam 4
MAT 250, Spring 2013 — D. Ivanšić

Name: ____________________

Show all your work!

Differentiate and simplify where appropriate:

1. (3pts) $\dfrac{d}{dx}\, e^{\cos x} =$

2. (4pts) $\dfrac{d}{dx}\, \ln(\sec x + 1) =$

3. (6pts) $\dfrac{d}{du}\, \dfrac{u+1}{3^u} =$

4. (7pts) $\dfrac{d}{dt}\, \ln \sqrt[6]{\dfrac{t^2 - 4t + 1}{3t^{-1}}} =$

5. (8pts) $\dfrac{d}{dx}\left(\dfrac{1}{2}\arcsin x + \dfrac{1}{2}x\sqrt{1-x^2}\right) =$

6. (10pts) Use logarithmic differentiation to find the derivative of $y = (x^2 - 4x + 7)^{\sin x}$.

Find the limits. Use L'Hospital's rule where appropriate.

7. (6pts) $\lim_{x \to 0^+} \arctan\left(\frac{x+2}{x}\right) =$

8. (8pts) $\lim_{x \to 0} \frac{\sin x - x}{x^3} =$

9. (6pts) $\lim_{x \to \infty} xe^{-x} =$

10. (10pts) $\lim_{x \to \infty} (1 + x^2)^{\frac{1}{x}} =$

11. (10pts) Find the critical points of the function $f(x) = x^{\frac{1}{3}}(x^2 + 9x)$.

12. (12pts) Let $f(x) = (x^2 - 3x - 3)e^x$. Find the absolute minimum and maximum values of f on the interval $[0, 4]$.

13. (10pts) Suppose θ is given implicitly as a function of x by $\tan\theta = \sqrt{x^2-1}$.
a) Use implicit differentiation to find θ'.
b) Using a trigonometric picture, express θ' only in terms of x.

Bonus. (10pts) Suppose that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Recall that we say $f(x)$ grows to infinity slower than $g(x)$ if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.

a) If $\lim_{x\to\infty} f(x) = \infty$, show that $\lim_{x\to\infty} \dfrac{\ln f(x)}{f(x)} = 0$.

b) If $\lim_{x\to\infty} f(x) = \infty$, show that there exists a function that grows to infinity slower than $f(x)$. This means there does not exist a “slowest-growing” function to infinity.

Calculus 1 — Exam 5 **MAT 250, Spring 2013 — D. Ivanšić**

Name: ______________________________

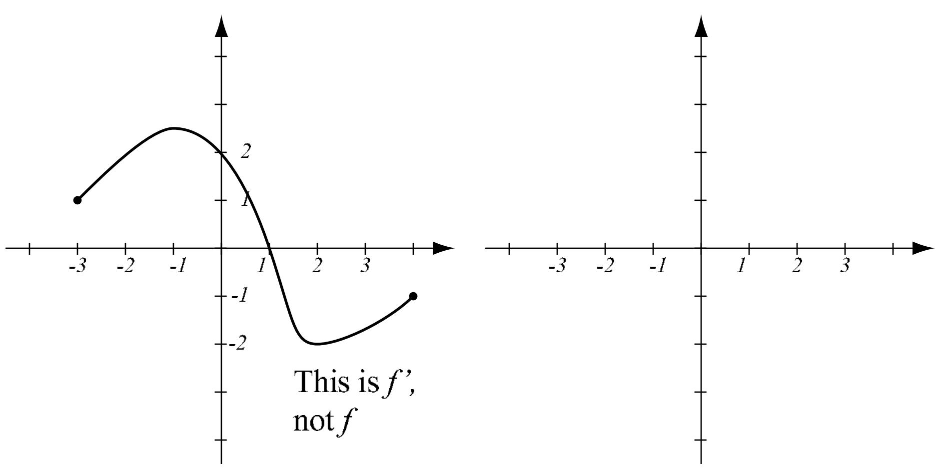
Show all your work!

1. (8pts) A function f is continuous and differentiable on $(0, \infty)$ and $-1 \leq f'(x) \leq 3$ for all x in $(0, \infty)$. If $f(3) = 6$, use the Mean Value Theorem to find the smallest and largest possible values for $f(8)$.

2. (14pts) Consider $f(x) = \sqrt{x}$ on the interval $[4, 9]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

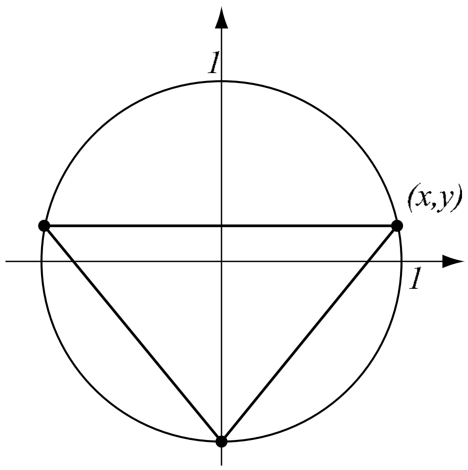
3. (14pts) Let $f(x) = \dfrac{x^2}{4} + \cos x$, $0 \le x \le 2\pi$. Find the intervals of concavity and points of inflection for f.

4. (14pts) Let f be continuous on $[-3, 4]$. The graph of its derivative f' is drawn below. Use the graph to answer (sign charts may help):
a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?
b) What are the intervals of concavity of f? Where does f have inflection points?
c) Use the information gathered in a) and b) to draw one possible graph of f at right.



5. (28pts) Let $f(x) = (x^2 + 1)e^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

6. (22pts) A triangle is inscribed into a circle of radius 1 so that one of its vertices is $(0, -1)$ and the side opposite this vertex is parallel to the x-axis.
a) In the picture, draw two more triangles that satisfy the requirements.
b) Among all such triangles, find the one with the maximal area. Verify that the one you found does, indeed, have maximal area.



Bonus. (10pts) Use calculus to show that $2 \arcsin x = \arccos(1 - 2x^2)$ for all x in $[0, 1]$. *(Hint: what do we know if derivatives of two functions are equal?)*

Calculus 1 — Exam 6
MAT 250, Spring 2013 — D. Ivanšić

Name: ____________________

Show all your work!

Find the following antiderivatives.

1. (3pts) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx =$

2. (3pts) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx =$

3. (4pts) $\displaystyle\int \sec(4\theta)\tan(4\theta)\,d\theta =$

4. (7pts) $\displaystyle\int t^2(t-\sqrt[4]{t})\,dt =$

5. (8pts) Find $f(x)$ if $f'(x) = x^{\frac{2}{3}} + \frac{7}{x}$ and $f(1) = 6$.

6. (6pts) Write using sigma notation:

$$\frac{1}{1} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{101}} =$$

7. (16pts) Find $\int_{-1}^{6} x - 3\, dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

8. (16pts) The function $f(x) = 4 - x^2$, $0 \le x \le 2$ is given.
a) Write down the expression R_4 that estimates the area under this curve using four approximating rectangles and right endpoints. Then evaluate the expresssion.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does R_4 represent? Does it over- or underestimate the area under the curve?

Use the substitution rule in the following integrals:

9. (9pts) $\int (x-4)\cos(x^2-8x+4)\,dx =$

10. (10pts) $\int_0^1 \frac{e^x+1}{(e^x+x)^2}\,dx =$

11. (10pts) The rate at which a deer population is growing is $2+\frac{t}{4}$ deer per day.
a) Use the Net Change Theorem to find how much the population has grown in 8 days.
b) If there were initially 107 deer, how many were there after 8 days?

12. (8pts) Show that $2 \leq \int_{-1}^{1} e^{x^2}\, dx \leq 2e$ **without** evaluating the integral.

Bonus. (10pts) A car is traveling at velocity 3 meters per second when it starts accelerating at constant acceleration. If it has traveled 108 meters during the 6 seconds that it accelerated, what is its acceleration?

Calculus 1 — Final Exam
MAT 250, Spring 2013 — D. Ivanšić

Name:

Show all your work!

1. (14pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

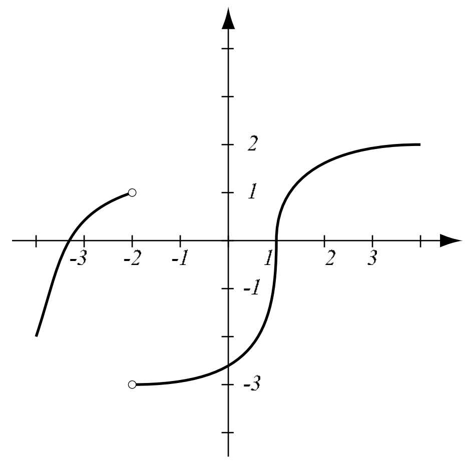
$\lim_{x\to -2^-} f(x) =$

$\lim_{x\to -2^+} f(x) =$

$\lim_{x\to -2} f(x) =$

$f(-2) =$

$\lim_{x\to 0} f(x) =$



List points where f is not continuous and explain why.

List points where f is not differentiable and explain why.

2. (8pts) Write the equation of the tangent line to the curve $y = \dfrac{x^2+1}{x-7}$ at point $\left(3, -\frac{5}{2}\right)$.

3. (10pts) Use logarithmic differentiation to find the derivative of $y = x^{\sqrt{x}}$.

Find the following limits algebraically. Do not use L'Hospital's rule.

4. (5pts) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}} =$

5. (5pts) $\displaystyle\lim_{x\to 5^-} \frac{x^2-2x}{x-5} =$

6. (8pts) Use L'Hospital's rule to find the limit: $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-x^2}{x^4} =$

7. (10pts) Use implicit differentiation to find y'.

$x\ln x + y\ln y = x^2 + y^2$

8. (12pts) Estimate $\sqrt[3]{8.6}$ using linear approximation by doing the following:
a) Write the linearization of the appropriate function at the appropriate point.
b) Use the linearization to estimate $\sqrt[3]{8.6}$ and compare it to the calculator-given value 2.0488.

9. (10pts) Let $f(x) = \cos^2 x - 2\sin^2 x$. Find the absolute minimum and maximum values of f on the interval $\left[\frac{\pi}{4}, \pi\right]$.

10. (25pts) Let $f(x) = xe^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

11. (8pts) Find $f(x)$ if $f'(x) = \dfrac{4}{x^2} - x^{\frac{3}{2}}$ and $f(4) = 3$.

12. (12pts) Consider the integral $\int_0^5 x^2 - 2x\,dx$.
a) Use a picture to determine whether this integral is positive or negative.
b) Evaluate the integral and verify your conclusion from a).

13. (9pts) Use substitution to find $\displaystyle\int \frac{3x^2 - 8x}{x^3 - 4x^2 + 7}\,dx =$

14. (14pts) A cylindrical tank with flexible sides contains 45m^3 of water (the tank is not full). The radius of the tank is shrinking at the rate of 0.1 meters per minute. How fast is the water level rising when the radius is 3 meters? Recall the volume of a cylinder is given by $V =$ area of base $\times$ height.

Bonus. (15pts) A car is traveling at velocity 3 meters per second when it starts accelerating at constant acceleration. Suppose it has traveled 108 meters during the 6 seconds that it accelerated.

a) What is its acceleration?

b) If after 6 seconds, the car stops accelerating and maintains a steady velocity, how long does it need, from the time it started accelerating, to cover 1000 meters ?