Calculus 1 — Final Exam
MAT 250, Spring 2013 — D. Ivanšić

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Show all your work!

1. (14pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

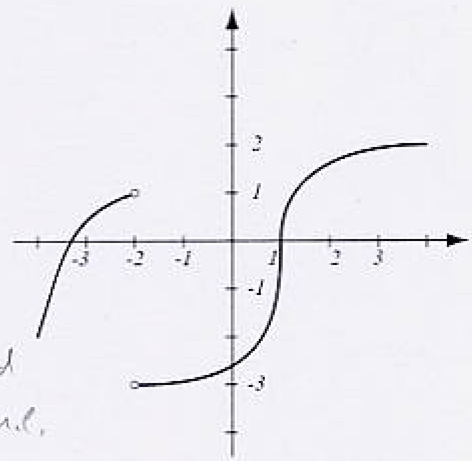
$\lim_{x\to -2^-} f(x) =$ 1

$\lim_{x\to -2^+} f(x) =$ -3

$\lim_{x\to -2} f(x) =$ d.n.e, one sided limits not equal

$f(-2) =$ is not defined

$\lim_{x\to 0} f(x) =$ -2.6



List points where f is not continuous and explain why.

Not cont. at $x=-2$, $f(-2)$ not defined and $\lim_{x\to -2} f(x)$ d.n.e.

List points where f is not differentiable and explain why.

$x=1$, vertical tangent line

$x=-2$, not even continuous at this point

2. (8pts) Write the equation of the tangent line to the curve $y = \dfrac{x^2+1}{x-7}$ at point $(3, -\frac{5}{2})$.

$$y' = \frac{2x(x-7)-(x^2+1)}{(x-7)^2} = \frac{x^2-14x-1}{(x-7)^2}$$

$$y'(3) = \frac{9-42-1}{16} = -\frac{34}{16} = -\frac{17}{8}$$

$$y-\left(-\frac{5}{2}\right) = -\frac{17}{8}(x-3)$$

$$y = -\frac{17}{8}x + \frac{51}{8} - \frac{5}{2}$$

$$= -\frac{17}{8}x + \frac{21}{8}$$

3. (10pts) Use logarithmic differentiation to find the derivative of $y = x^{\sqrt{x}}$.

$$y = x^{\sqrt{x}}$$

$$\ln y = \ln x^{\sqrt{x}}$$

$$\ln y = \sqrt{x}\ln x \quad \Big| \frac{d}{dx}$$

$$\frac{1}{y}\cdot y' = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} \quad \Big| \cdot y$$

$$y' = x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

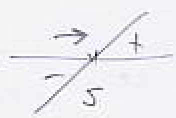
$$= \frac{x^{\sqrt{x}}}{2\sqrt{x}}\left(\ln x + 2\right)$$

Find the following limits algebraically. Do not use L'Hospital's rule.

4. (5pts) $\lim\limits_{x\to3}\dfrac{x^2-9}{\sqrt{x}-\sqrt{3}} = \lim\limits_{x\to3}\dfrac{x^2-9}{\sqrt{x}-\sqrt{3}}\cdot\dfrac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim\limits_{x\to3}\dfrac{\cancel{(x-3)}(x+3)(\sqrt{x}+\sqrt{3})}{\cancel{x-3}}$

$= \lim\limits_{x\to3}(x+3)(\sqrt{x}+\sqrt{3}) = 6\cdot2\sqrt{3} = 12\sqrt{3}$

5. (5pts) $\lim\limits_{x\to5^-}\dfrac{x^2-2x}{x-5} = \dfrac{5^2-2\cdot5}{0^-} = \dfrac{15}{0^-} = -\infty$



When $x\to5^-$

$x-5<0$

$\dfrac{15}{\text{small neg}}$ = large neg.

$1-0-1=0$

6. (8pts) Use L'Hospital's rule to find the limit: $\lim\limits_{x\to0}\dfrac{e^{x^2}-x^2-1}{x^4} \overset{L'H}{=} \lim\limits_{x\to0}\dfrac{e^{x^2}\cdot2x-2x}{4x^3}$

0

$1-1=0$

$= \lim\limits_{x\to0}\dfrac{2x(e^{x^2}-1)}{4x^3} = \lim\limits_{x\to0}\dfrac{e^{x^2}-1}{2x^2} \overset{L'H}{=} \lim\limits_{x\to0}\dfrac{e^{x^2}\cdot2x}{4x} = \lim\limits_{x\to0}\dfrac{e^{x^2}}{2} = \dfrac{1}{2}$

0

7. (10pts) Use implicit differentiation to find y'.

$x\ln x + y\ln y = x^2+y^2 \quad \Big|\ \dfrac{d}{dx}$

$1\cdot\ln x + x\cdot\dfrac{1}{x} + y'\ln y + y\cdot\dfrac{1}{y}\cdot y' = 2x+2yy'$

$\ln x + 1 + y'\ln y + y' = 2x+2yy' \quad \Big|\ \begin{matrix}-\ln x - 1\\ -2yy'\end{matrix}$

$y'\ln y + y' - 2yy' = 2x - \ln x - 1$

$y'(\ln y + 1 - 2y) = 2x - \ln x - 1$

$y' = \dfrac{2x-\ln x-1}{\ln y+1-2y}$

10. (25pts) Let $f(x) = xe^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

$$f'(x) = 1\cdot e^x + xe^x = (x+1)e^x$$

$$f''(x) = 1\cdot e^x + (x+1)e^x = (x+2)e^x$$

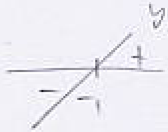
a) $(x+1)\underbrace{e^x}_{>0 \text{ always}} = 0$

$x+1=0$

$x=-1$

		-1	
$x+1$	$-$	0	$+$
f	$\searrow$	loc. min	$\nearrow$

Sign of f' = sign of $x+1$



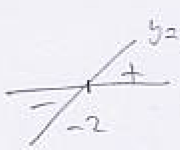
b) $(x+2)\underbrace{e^x}_{>0} = 0$

$x+2=0$

$x=-2$

		-2	
$x+2$	$-$	0	$+$
f	CD	IP	CU

Sign of f'' = Sign of $x+2$

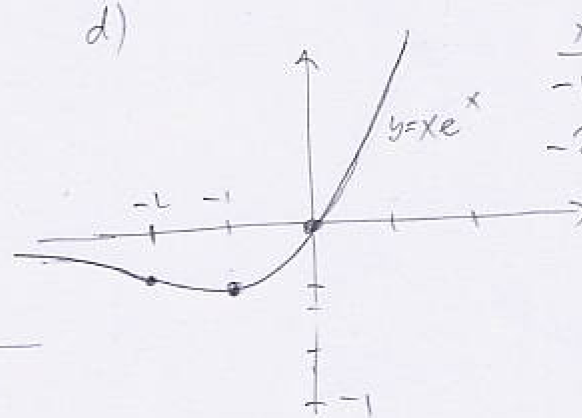


c) $\lim_{x\to\infty} xe^x = \infty\cdot\infty = \infty$

$\lim_{x\to-\infty} \underset{-\infty\cdot 0}{xe^x} = \lim_{x\to-\infty} \frac{x}{e^{-x}} \overset{L'H}{=}$

$= \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0$

d)



x	xe^x
-1	$-e^{-1} \approx -\frac{1}{3}$
-2	$-2e^{-2} \approx -\frac{2}{9}$

8. (12pts) Estimate $\sqrt[3]{8.6}$ using linear approximation by doing the following:
a) Write the linearization of the appropriate function at the appropriate point.
b) Use the linearization to estimate $\sqrt[3]{8.6}$ and compare it to the calculator-given value 2.0488.

a) $f(x) = \sqrt[3]{x}$, linearize around $a = 8$ $\quad f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$L(x) = f(8) + f'(8)(x-8)$

$= 2 + \cdot\, 8^{-\frac{2}{3}}(x-8)$

$= 2 + \frac{1}{12}(x-8)$

b) $L(8.6) = 2 + \frac{1}{12}(8.6-8) = 2 + \frac{0.6}{12} = 2 + \frac{0.1}{2} = 2.05$, within 0.0012 of calculator value

$= 2 + \frac{0.6}{12} =$

9. (10pts) Let $f(x) = \cos^2 x - 2\sin^2 x$. Find the absolute minimum and maximum values of f on the interval $\left[\frac{\pi}{4}, \pi\right]$.

$f'(x) = 2\cos x(-\sin x) - 4\sin x\cos x$

$= -6\sin x\cos x$

$\sin x\cos x = 0$

$\sin x = 0$ or $\cos x = 0$ a $\left[\frac{\pi}{4}, \pi\right]$

$x = \pi \qquad x = \frac{\pi}{2}$

x	$\cos^2 x - 2\sin^2 x$	
$\frac{\pi}{2}$	$0 - 2 = -2$	abs min
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}\right)^2 - 2\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} - 1 = -\frac{1}{2}$	
π	$(-1)^2 - 0 = 1$	abs. max

11. (8pts) Find $f(x)$ if $f'(x) = \dfrac{4}{x^2} - x^{\frac{3}{2}}$ and $f(4) = 3$.

$$f'(x) = 4x^{-2} - x^{\frac{3}{2}}$$

$$f(x) = 4\frac{x^{-1}}{-1} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = -\frac{4}{x} - \frac{2}{5}x^{\frac{5}{2}} + C$$

$$3 = f(4) = -\frac{4}{4} - \frac{2}{5}4^{\frac{5}{2}} + C$$

$$3 = -1 - \frac{2}{5}\cdot 32 + C$$

$$C = 4 + \frac{64}{5} = \frac{84}{5}$$

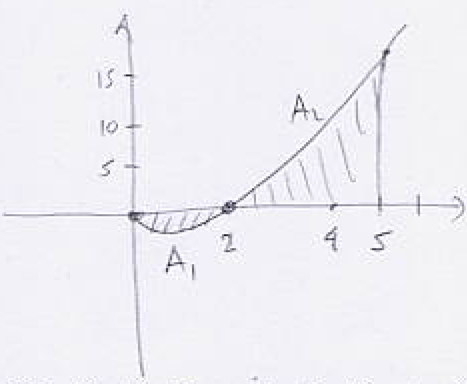
$$f(x) = -\frac{4}{x} - \frac{2}{5}x^{\frac{5}{2}} + \frac{84}{5}$$

12. (12pts) Consider the integral $\int_0^5 x^2 - 2x\,dx$.
a) Use a picture to determine whether this integral is positive or negative.
b) Evaluate the integral and verify your conclusion from a).

a) $x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$



$\int_0^5 x^2 - 2x\,dx = -A_1 + A_2$, which is positive, since $A_2 > A_1$

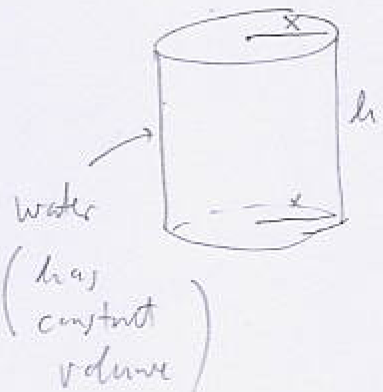
b) $$\int_0^5 x^2 - 2x\,dx = \left(\frac{x^3}{3} - x^2\right)\Big|_0^5 = \left(\frac{125}{3} - 25\right) - 0$$

$$= \frac{125 - 75}{3} = \frac{50}{3}, \text{ positive}$$

13. (9pts) Use substitution to find $\displaystyle\int \frac{3x^2 - 8x}{x^3 - 4x^2 + 7}\,dx =$ $\left[\begin{array}{l} u = x^3 - 4x^2 + 7 \\ du = 3x^2 - 8x \end{array}\right]$

$$= \int \frac{du}{u} = \ln|u| = \ln|x^3 - 4x^2 + 7| + C$$

14. (14pts) A cylindrical tank with flexible sides contains 45m^3 of water (the tank is not full). The radius of the tank is shrinking at the rate of 0.1 meters per minute. How fast is the water level rising when the radius is 3 meters? Recall the volume of a cylinder is given by V = area of base $\times$ height.



Know: $x' = -0.1$ meters/min

Need: h', when $x = 3$

$$V = \pi x^2 h \quad \Big| \frac{d}{dt}$$

$$0 = \pi(2xx'h + x^2h')$$

$$x^2h' = -2xx'h$$

$$h' = -\frac{2xx'h}{x^2} = -\frac{2x'h}{x}$$

When $x = 3$

$$45 = \pi \cdot 9 \cdot h$$

$$h = \frac{45}{9\pi} = \frac{5}{\pi}$$

$$h' = -\frac{2\cdot(-0.1)\cdot\frac{5}{\pi}}{3}$$

$$= \frac{10}{3\pi} \text{ m/min}$$

Bonus. (15pts) A car is traveling at velocity 3 meters per second when it starts accelerating at constant acceleration. Suppose it has traveled 108 meters during the 6 seconds that it accelerated.

a) What is its acceleration?

b) If after 6 seconds, the car stops accelerating and maintains a steady velocity, how long does it need, from the time it started accelerating, to cover 1000 meters ?

Let A = constant acceleration

a) $a(t) = A$, $v(0) = 3$ m/s

$v(t) = At + C$

$3 = v(0) = A \cdot 0 + C$, so $C = 3$

$v(t) = At + 3$

$s(t) = A \cdot \frac{1}{2}t^2 + 3t + D$

$0 = s(0) = 0 + 0 + D$ so $D = 0$

$s(t) = \frac{A}{2}t^2 + 3t$

$108 = s(6) = \frac{A}{2} \cdot 36 + 18$

$108 = 18A + 18$

$90 = 18A$

$A = 5$

b) $v(6) = 5 \cdot 6 + 3 = 33$ m/s

$s(6) = 108$, needs to cover $1000 - 108$ m at constant rate 33, $t = \frac{1000-108}{33} = \frac{892}{33}$ s