Calculus 1 — Exam 5
MAT 250, Spring 2013 — D. Ivanšić

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1. (8pts) A function f is continuous and differentiable on $(0, \infty)$ and $-1 \leq f'(x) \leq 3$ for all x in $(0, \infty)$. If $f(3) = 6$, use the Mean Value Theorem to find the smallest and largest possible values for $f(8)$.

By Mean Value theorem, there is a c in (3,8) such that

$f'(c) = \frac{f(8)-f(3)}{8-3}$. Since $-1 \leq f'(c) \leq 3$, we have:

$$-1 \leq \frac{f(8)-f(3)}{8-3} \leq 3$$

$$-1 \leq \frac{f(8)-6}{5} \leq 3 \quad | \cdot 5$$

$$-5 \leq f(8)-6 \leq 15$$

$$\boxed{1 \leq f(8) \leq 21}$$

2. (14pts) Consider $f(x) = \sqrt{x}$ on the interval $[4, 9]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

a) $\sqrt{x}$ is continuous and differentiable on $(0, \infty)$, so especially on $[4, 9]$

b) $\frac{\sqrt{9}-\sqrt{4}}{9-4} = \frac{3-2}{5} = \frac{1}{5}$

By MVT, there is a c in (4,9) st. $f'(c) = \frac{1}{5}$

$$\frac{1}{2\sqrt{x}} = \frac{1}{5}$$

$$5 = 2\sqrt{x}$$

$$\sqrt{x} = \frac{5}{2} \quad |^2$$

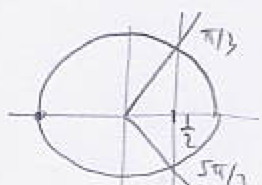
$x = \frac{25}{4} = 6\frac{1}{4}$, which is in $(4, 9)$

3. (14pts) Let $f(x) = \dfrac{x^2}{4} + \cos x$, $0 \le x \le 2\pi$. Find the intervals of concavity and points of inflection for f.

$f'(x) = \dfrac{2x}{4} - \sin x$

$f''(x) = \dfrac{1}{2} - \cos x$

$\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}, \frac{5\pi}{3}$

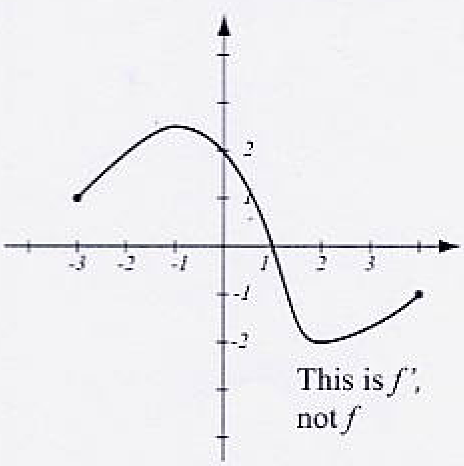


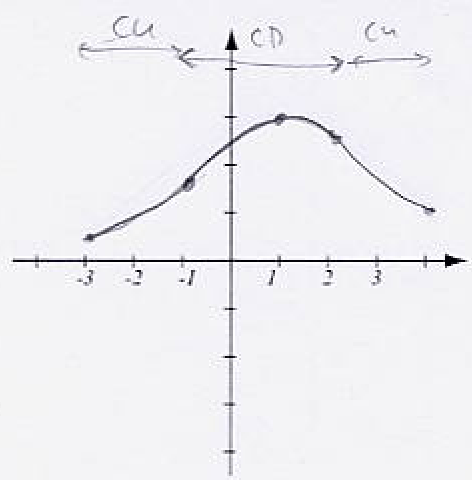
x	0		$\frac{\pi}{3}$		$\frac{5\pi}{3}$		2π
$f''(x)$		$-$	0	$+$	0	$-$	
$f(x)$		CD	IP	CU	IP	CD	

points of inflection

x	$f''(x)$
0	$\frac{1}{2} - 1 = -\frac{1}{2}$
π	$\frac{1}{2} - (-1) = \frac{3}{2}$
2π	$-\frac{1}{2}$

4. (14pts) Let f be continuous on $[-3, 4]$. The graph of its derivative f' is drawn below. Use the graph to answer (sign charts may help):
a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?
b) What are the intervals of concavity of f? Where does f have inflection points?
c) Use the information gathered in a) and b) to draw one possible graph of f at right.





a) f incr / decr $\Leftrightarrow$ $f' > 0$ / < 0

f incr on $(-3, 1)$
f decr on $(1, 4)$

local max at $x = 1$

b) f CU / CD $\Leftrightarrow$ f' incr / decr.

f CU on $(-3, -1) \cup (2, 4)$
f CD on $(-1, 2)$

Inflection pt. at $x = -1$, $x = 2$

5. (28pts) Let $f(x) = (x^2+1)e^x$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

$$f'(x) = 2xe^x + (x^2+1)e^x$$
$$= (x^2+2x+1)e^x = (x+1)^2e^x$$
$$f''(x) = (2x+2)e^x + (x^2+2x+1)e^x$$
$$= (x^2+4x+3)e^x$$

c) $\lim\limits_{x\to\infty} (x^2+1)e^x = \infty\cdot\infty = \infty$

$\infty \cdot 0$

$$\lim_{x\to-\infty} (x^2+1)e^x = \lim_{x\to-\infty} \frac{x^2+1}{e^{-x}} \quad \frac{\to\infty}{\to\infty}$$

$$\overset{L'H}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{L'H}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{\infty} = 0$$

($2x \to -\infty$, $-e^{-x} \to -\infty$)

a) Crit. pts: $f'(x)=0$ or d.n.e. never

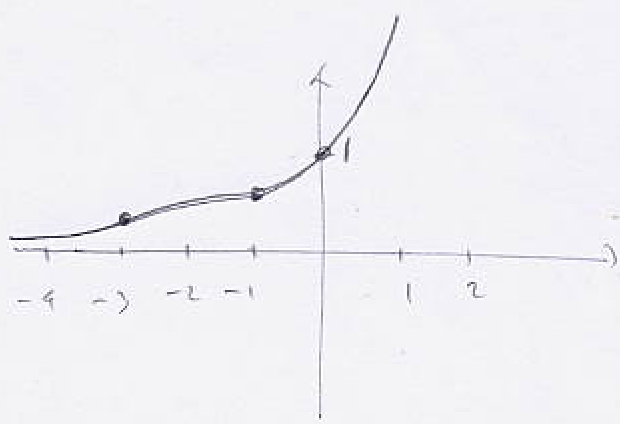
$(x+1)^2 \underbrace{e^x}_{>0} = 0$

$x=-1$

Since both $(x+1)^2$, e^x are positive $f'(x)$ is always positive, so f is always increasing.

d)

x	$(x^2+1)e^x$
-1	$2e^{-1} \approx \frac{2}{3}$
-3	$10e^{-3} \approx \frac{10}{27}$



b) 2nd order crit. pts:

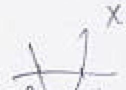
$(x^2+4x+3)e^x = 0$ or $\underset{\text{never}}{\text{d.n.e.}}$

$(x+3)(x+1)e^x = 0$

$x = -1, -3$

same sign as $f''(x)$ since $e^x > 0$

x^2+4x+3

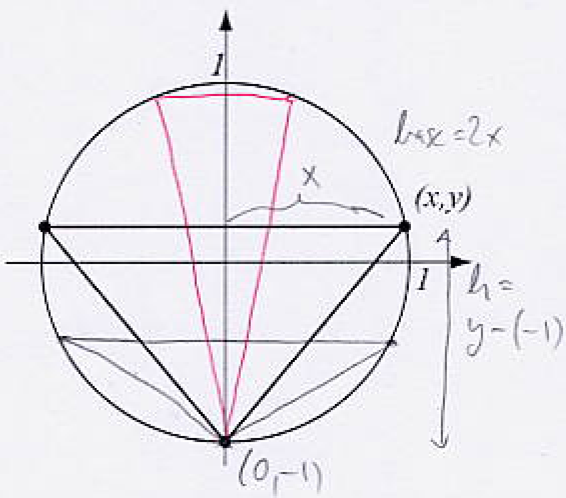


		-3		-1	
x^2+4x+3	$+$	0	$-$	0	$+$
f	CU	IP	CD	IP	CU

6. (22pts) A triangle is inscribed into a circle of radius 1 so that one of its vertices is $(0, -1)$ and the side opposite this vertex is parallel to the x-axis.

a) In the picture, draw two more triangles that satisfy the requirements.

b) Among all such triangles, find the one with the maximal area. Verify that the one you found does, indeed, have maximal area.



$A = \frac{1}{2} b \cdot h = \frac{1}{2} \cdot 2x \cdot (y+1) = x(y+1)$

Since $x^2 + y^2 = 1$, $x = \sqrt{1-y^2}$, $A = \sqrt{1-y^2}\,(y+1)$

Since A^2 is maximal when A is, consider A^2

Job: Maximize $f(y) = \left(\sqrt{1-y^2}(y+1)\right)^2$ on $[-1, 1]$

$= (1-y^2)(y+1)^2$

$f'(y) = -2y(y+1)^2 + (1-y^2)2(y+1)$

$= 2(y+1)(-y(y+1) + 1 - y^2)$

$= 2(y+1)(-2y^2 - y + 1)$

Crit. pts: $y = -1$ $\quad 2y^2 + y - 1 = 0$

$y = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}$

y	$(1-y^2)(y+1)^2$	
-1	0	
$\frac{1}{2}$	$(1-\frac{1}{4})(\frac{3}{2})^2 = \frac{3}{4} \cdot \frac{9}{4} = \frac{27}{16}$	abs. max
1	0	

Bonus. (10pts) Use calculus to show that $2 \arcsin x = \arccos(1 - 2x^2)$ for all x in $[0, 1]$. *(Hint: what do we know if derivatives of two functions are equal?)*

$(2 \arcsin x)' = \frac{2}{\sqrt{1-x^2}}$

$(\arccos(1-2x^2))' = -\frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot (-4x) = \frac{4x}{\sqrt{1-(1-4x^2+4x^4)}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2(1-x^2)}}$

$\left[\begin{matrix}\sqrt{x^2} = x \\ \text{since } x > 0\end{matrix}\right] = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$ equal. When $f'(x) = g'(x)$, then $f(x) = g(x) + C$

so $\quad 2 \arcsin x = \arccos(1-2x^2) + C \quad$ plug in $x = 0$

$2 \arcsin 0 = \arccos 1 + C \quad 0 = 0 + C$ so $C = 0$ and the two functions are equal.