Calculus 1 — Exam 4 MAT 250, Spring 2013 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (3pts) $\frac{d}{dx} e^{\cos x} = e^{\cos x} \cdot (-\sin x) = -\sin x\, e^{\cos x}$

2. (4pts) $\frac{d}{dx} \ln(\sec x + 1) = \frac{1}{\sec x + 1}(\sec x \cdot \tan x) = \frac{\sec x \tan x}{\sec x + 1} \cdot \frac{\cos x}{\cos x} = \frac{\tan x}{1+\cos x}$

3. (6pts) $\frac{d}{du} \frac{u+1}{3^u} = \frac{1 \cdot 3^u - (u+1)\ln 3 \cdot 3^u}{(3^u)^2} = \frac{\cancel{3^u}(1-(u+1)\ln 3)}{(3^u)^2} = \frac{-u\ln 3 - \ln 3 + 1}{3^u}$

4. (7pts) $\frac{d}{dt} \ln \sqrt[6]{\frac{t^2-4t+1}{3t-1}} = \frac{d}{dt} \ln\left(\frac{t^2-4t+1}{3t-1}\right)^{\frac{1}{6}} = \frac{d}{dt}\frac{1}{6}\left(\ln(t^2-4t+1) - \ln(3t-1)\right)$

$6t^2-14t+4-3t^2+12t-3$

$= \frac{1}{6}\frac{1}{t^2-4t+1}\cdot(2t-4) - \frac{1}{3t-1}\cdot 3 = \frac{1}{6}\cdot\frac{(2t-4)(3t-1)-3(t^2-4t+1)}{(t^2-4t+1)(3t-1)} = \frac{3t^2-2t+1}{6(t^2-4t+1)(3t-1)}$

5. (8pts) $\frac{d}{dx}\left(\frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}\right) = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\left(1\cdot\sqrt{1-x^2} + x\frac{1}{2\sqrt{1-x^2}}(-2x)\right)$

$= \frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) = \frac{1}{2}\frac{1+(1-x^2)-x^2}{\sqrt{1-x^2}} = \frac{2(1-x^2)}{2\sqrt{1-x^2}} = \sqrt{1-x^2}$

6. (10pts) Use logarithmic differentiation to find the derivative of $y = (x^2-4x+7)^{\sin x}$.

$y = (x^2-4x+7)^{\sin x}$

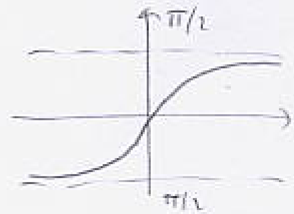
$\ln y = \sin x \ln(x^2-4x+7) \quad \Big|\frac{d}{dx}$

$\frac{1}{y}\cdot y' = \cos x \ln(x^2-4x+7) + \sin x \frac{1}{x^2-4x+7}\cdot(2x-4) \quad \Big|\cdot y$

$y' = \left(\cos x \ln(x^2-4x+7) + \frac{(2x-4)\sin x}{x^2-4x+7}\right)(x^2-4x+7)^{\sin x}$

Find the limits. Use L'Hospital's rule where appropriate.

7. (6pts) $\lim_{x\to 0^+} \arctan\left(\frac{x+2}{x}\right) = \arctan\frac{2}{0+} = \arctan\infty = \frac{\pi}{2}$



8. (8pts) $\lim_{x\to 0} \frac{\overset{\to 0}{\sin x - x}}{\underset{\to 0}{x^3}} \overset{L'H}{=} \lim_{x\to 0} \frac{\overset{\to 1-1=0}{\cos x - 1}}{\underset{\to 0}{3x^2}} \overset{L'H}{=} \lim_{x\to 0} \frac{\overset{\to 0}{-\sin x}}{\underset{\to 0}{6x}} \overset{L'H}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

9. (6pts) $\underset{\infty\cdot 0}{\lim_{x\to\infty} xe^{-x}} = \lim_{x\to\infty} \frac{\overset{\to\infty}{x}}{\underset{\to\infty}{e^x}} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

10. (10pts) $\lim_{x\to\infty} (1+x^2)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$ $\qquad \infty^0$

$y = (1+x^2)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(1+x^2)$

$\lim_{x\to\infty} \frac{\overset{\to\infty}{\ln(1+x^2)}}{\underset{\to\infty}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}\cdot 2x}{1} = \lim_{x\to\infty} \frac{\overset{\to\infty}{2x}}{\underset{\to\infty}{1+x^2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{2x} = \frac{1}{\infty} = 0$

11. (10pts) Find the critical points of the function $f(x) = x^{\frac{1}{3}}(x^2+9x)$.

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x^2+9x) + x^{\frac{1}{3}}(2x+9) = x^{-\frac{2}{3}}\left(\frac{1}{3}(x^2+9x) + x(2x+9)\right) = x^{-\frac{2}{3}}\left(\frac{7}{3}x^2 + 12x\right)$

$= x^{-\frac{2}{3}} \cdot x\left(\frac{7}{3}x + 12\right) = x^{\frac{1}{3}}\left(\frac{7}{3}x+12\right)$

$f'(x) = 0$

$\frac{7}{3}x + 12 = 0$ or $x^{\frac{1}{3}} = 0$

$f'(x)$ is always defined.

$\frac{7}{3}x = -12$ $\quad$ $x = 0$

$x = -\frac{36}{7}$

12. (12pts) Let $f(x) = (x^2 - 3x - 3)e^x$. Find the absolute minimum and maximum values of f on the interval $[0, 4]$.

Critical points:

$f'(x) = (2x-3)e^x + (x^2-3x-3)e^x$

$= e^x(2x-3+x^2-3x-3)$

$= e^x(x^2-x-6)$

$\underbrace{e^x}_{\text{always} > 0}(x^2-x-6) = 0$

$(x-3)(x+2) = 0$

$x = 3, \underset{\text{not in interval}}{-2}$

Check values:

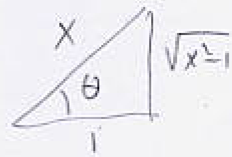
x	$(x^2-3x-3)e^x$	
3	$-3e^3$	abs min value
0	-3	
4	$(16-12-3)e^4 = e^4$	abs. max value

13. (10pts) Suppose θ is given implicitly as a function of x by $\tan\theta = \sqrt{x^2-1}$.
a) Use implicit differentiation to find θ'.
b) Using a trigonometric picture, express θ' only in terms of x.

$$\tan\theta = \sqrt{x^2-1} \quad \Big| \; \frac{d}{dx}$$

$$\sec^2\theta\,\theta' = \frac{1}{2\sqrt{x^2-1}} \cdot 2x$$

$$\theta' = \frac{x}{\sec^2\theta\sqrt{x^2-1}} = \frac{\overset{1}{\cancel{x}}}{x^{\cancel{2}}\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$$



$$\left(\sqrt{x^2-1}\right)^2 + 1^2 = x^2-1+1 = x^2$$

$$\cos\theta = \frac{1}{x}$$

$$\sec\theta = x$$

Bonus. (10pts) Suppose that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Recall that we say $f(x)$ grows to infinity slower than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

a) If $\lim_{x\to\infty} f(x) = \infty$, show that $\lim_{x\to\infty} \frac{\ln f(x)}{f(x)} = 0$.

b) If $\lim_{x\to\infty} f(x) = \infty$, show that there exists a function that grows to infinity slower than $f(x)$. This means there does not exist a "slowest-growing" function to infinity.

$\to \ln\infty = \infty$

a) $$\lim_{x\to\infty} \frac{\ln f(x)}{\underset{\to\infty}{f(x)}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{f(x)}\cdot \cancel{f'(x)}}{\cancel{f'(x)}} = \lim_{x\to\infty} \frac{1}{f(x)} = \frac{1}{\infty} = 0$$

b) In a) we showed $\lim_{x\to\infty} \frac{\ln f(x)}{f(x)} = 0$ so $\ln f(x)$ grows slower than $f(x)$.