Calculus 1 — Exam 3
MAT 250, Spring 2013 — D. Ivanšić

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1. (14pts) Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$.

$x^2 + 4xy + y^2 = 13$

$2x + 4(1\cdot y + x\cdot y') + 2yy' = 0 \quad |\div 2$

$x + 2y + 2xy' + yy' = 0$

$y'(2x+y) = -x - 2y$

$y' = -\dfrac{x+2y}{2x+y}$

$y'\big|_{(2,1)} = -\dfrac{2+2}{4+1} = -\dfrac{4}{5}$

Eq. of tangent line:

$y - 1 = -\frac{4}{5}(x-2)$

$y = -\frac{4}{5}x + \frac{8}{5} + 1$

$y = -\frac{4}{5}x + \frac{13}{5}$

2. (12pts) Use implicit differentiation to find y'.

$\sin x + \cos y = \dfrac{x}{y} \quad \Big|\dfrac{d}{dx}$

$\cos x - \sin y\, y' = \dfrac{1\cdot y - x\cdot y'}{y^2} \quad |\cdot y^2$

$y^2\cos x - y^2 \sin y\, y' = y - xy'$

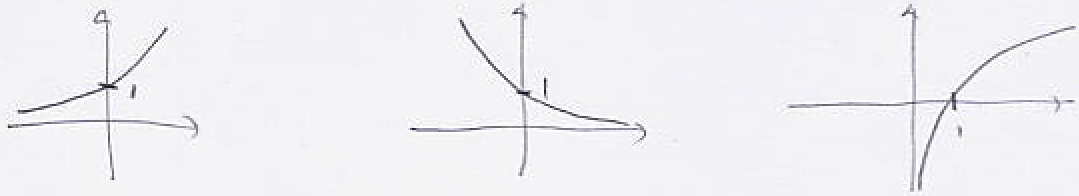
$y^2\cos x - y = y^2\sin y\, y' - xy'$

$y^2\cos x - y = (y^2\sin y - x)y'$

$y' = \dfrac{y^2\cos x - y}{y^2\sin y - x}$

3. (6pts) Sketch the graphs of the following functions.

$y = a^x$, $a > 1$ $\qquad$ $y = a^x$, $0 < a < 1$ $\qquad$ $y = \log_a x$, $a > 1$



Find the following limits algebraically. The graphs you drew above may help you.

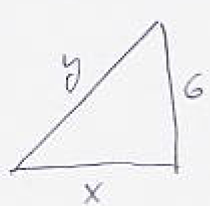
4. (2pts) $\lim_{x\to\infty} \left(\frac{1}{3}\right)^x = 0$ as can be seen from second graph

5. (6pts) $\lim_{x\to\infty} \frac{5^x + 4}{2 \cdot 5^x - 7} = \lim_{x\to\infty} \frac{\cancel{5^x}\left(1 + \frac{4}{5^x}\right)}{\cancel{5^x}\left(2 - \frac{7}{5^x}\right)} = \frac{1 + \frac{4}{\infty}}{2 - \frac{7}{\infty}} = \frac{1+0}{2-0} = \frac{1}{2}$

6. (6pts) $\lim_{x\to 0^+} e^{1-\frac{1}{x}} = e^{\lim_{x\to 0+} 1 - \frac{1}{x}} = e^{1-\frac{1}{0+}} = e^{1-\infty} = e^{-\infty} = 0$

from first graph

7. (14pts) One side of a right triangle is known to have length 6 feet. The other side has been measured to be 8 feet, with maximum error $\frac{1}{2}$ of an inch. Use differentials to estimate the maximum possible error, the relative error and the percentage error when computing the length of the hypothenuse of this triangle.



$x^2 + 36 = y^2$

$y = \sqrt{x^2+36}$

$dy = \frac{2x}{2\sqrt{x^2+36}} \cdot dx$

$dy = \frac{x}{\sqrt{x^2+36}}\, dx$

Put in $x = 8$, $dx = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$ ft

$dy = \frac{\cancel{8}}{\sqrt{64+36}} \cdot \frac{1}{\cancel{24}_3} = \frac{1}{10 \cdot 3} = \frac{1}{30}$ ft

$\frac{dy}{y} = \frac{\frac{1}{30}}{10} = \frac{1}{300}$

percentage $\frac{1}{300} = \frac{1}{3}\,\%$

8. (12pts) Estimate $\sqrt{4.5}$ using linear approximation by doing the following:
a) Write the linearization of the appropriate function at the appropriate point.
b) Use the linearization to estimate $\sqrt{4.5}$ and compare it to the calculator-given value 2.12132.

a) $f(x) = \sqrt{x}$ $\qquad L(x) = 2 + \frac{1}{4}(x-4) = \frac{1}{4}x + 1$

$a = 4,\ f(4) = 2$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(4) = \dfrac{1}{2\cdot\sqrt{4}} = \dfrac{1}{4}$

b) $L(4.5) = 2 + \frac{1}{4}\cdot 0.5 = 2 + \frac{1}{8} = 2.125$, less than 0.004 away from actual value

9. (12pts) Let $f(x) = x^2 - 10x + 20$, $x \leq 5$. Use the theorem on derivatives of inverses to find $(f^{-1})'(4)$.

$f(x) = x^2 - 10x + 20$

$f'(x) = 2x - 10$

$(f^{-1})'(4) = \dfrac{1}{f'(f^{-1}(4))} = \dfrac{1}{2\cdot f^{-1}(4) - 10} = \dfrac{1}{2\cdot 2 - 10} = -\dfrac{1}{6}$

$f^{-1}(4) = x$ such that $f(x) = 4$

$x^2 - 10x + 20 = 4$

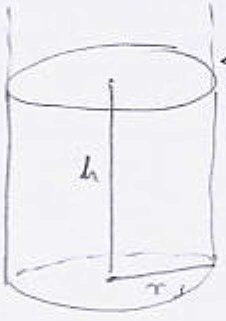
$x^2 - 10x + 16 = 0$

$(x-2)(x-8) = 0$

$x = 2, 8$

only this one in domain $x \leq 5$

10. (16pts) A cylindrical tank with flexible sides contains 45m^3 of water (the tank is not full). The radius of the tank is shrinking at the rate of 0.1 meters per minute. How fast is the water level rising when the radius is 3 meters? Recall the volume of a cylinder is given by V = area of base $\times$ height.



← water inside the tank, 45 m^3

Know: $r' = -0.1$ m/min

Need: h' when $r = 3$

$$\pi r^2 \cdot h = 45 \quad \Big| \frac{d}{dt}$$

$$\pi(2rr'h + r^2h') = 0 \quad \Big| \div \pi$$

$$r^2h' = -2rr'h$$

$$h' = -\frac{2rr'h}{r^2}$$

$$h' = -\frac{2r'h}{r}$$

When $r = 3$:

$$\pi \cdot 3^2 \cdot h = 45$$

so $h = \frac{45}{9\pi} = \frac{5}{\pi}$

Then. $h' = -\frac{2\cdot(-0.1)\cdot\frac{5}{\pi}}{3}$

$$= \frac{0.2\cdot 5}{3\pi} = \frac{1}{3\pi} \text{ m/min}$$

Bonus. (10pts) Find the formula for the inverse of the function in problem 9. Use it to find $(f^{-1})'$ and $(f^{-1})'(4)$. You should get the same answer as in problem 9.

$x^2 - 10x + 20 = y$ solve for x

$x^2 - 10x + 20 - y = 0$ quadratic equation in x

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4\cdot 1\cdot(20-y)}}{2}$$

$$= \frac{10 \pm \sqrt{100 - 80 + 4y}}{2}$$

$$= \frac{10 \pm \sqrt{20+4y}}{2} = \frac{10 \pm \sqrt{4(5+y)}}{2}$$

$$= \frac{2(5 \pm \sqrt{5+y})}{2} = 5 \pm \sqrt{5+y}$$

Since the values have to be ≤ 5,

$$f^{-1}(y) = 5 - \sqrt{5+y} \quad \Big| \frac{d}{dy}$$

$$(f^{-1})'(y) = -\frac{1}{2\sqrt{5+y}} \cdot 1 = -\frac{1}{2\sqrt{5+y}}$$

$(f^{-1})'(4) = -\frac{1}{2\sqrt{5+4}} = -\frac{1}{6}$, agrees with 9.