Calculus 1 — Exam 2
MAT 250, Spring 2013 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(4x^4 + \frac{4}{x^2} - x^2\sqrt{x} + \sqrt[4]{\pi}\right) = 16x^3 - 8x^{-3} + \frac{5}{2}x^{\frac{3}{2}} = 16x^3 - \frac{8}{x^3} + \frac{5}{2}x\sqrt{x}$

$4x^{-2}$ $\quad x^2x^{\frac{1}{2}}$ $\quad$ constant

$= x^{\frac{5}{2}}$

2. (5pts) $\frac{d}{du}(u^2 - 3u)\sqrt{4u-7} = (2u-3)\sqrt{4u-7} + (u^2-3u)\frac{1}{2\sqrt{4u-7}} \cdot 4$

$= (2u-3)\sqrt{4u-7} + \frac{2(u^2-3u)}{\sqrt{4u-7}} = \frac{1}{\sqrt{4u-7}}\left((2u-3)(4u-7) + 2(u^2-3u)\right) = \frac{10u^2 - 32u + 21}{\sqrt{4u-7}}$

more difficult, so didn't intend for you to go so far

3. (6pts) $\frac{d}{dx}\frac{x-4}{x^2-3x+1} = \frac{1\cdot(x^2-3x+1) - (x-4)(2x-3)}{(x^2-3x+1)^2} = \frac{x^2-3x+1 - (2x^2-11x+12)}{(x^2-3x+1)^2}$

$= \frac{-x^2+8x-11}{(x^2-3x+1)^2}$

4. (6pts) $\frac{d}{d\theta}(\cos^2\theta - \sin^2\theta) = 2\cos\theta\cdot(-\sin\theta) - 2\sin\theta\cos\theta$

$= -4\sin\theta\cos\theta$

5. (7pts) $\frac{d}{dx}\sqrt{\tan(x^4+x^2+1)} = \frac{1}{2\sqrt{\tan(x^4+x^2+1)}}\cdot\sec^2(x^4+x^2+1)\cdot\overbrace{(4x^3+2x)}^{2x(2x^2+1)}$

$= \frac{x(2x^2+1)\sec^2(x^4+x^2+1)}{\sqrt{\tan(x^4+x^2+1)}}$

6. (8pts) Let $h(x) = f(x)\sin x$. Find the general expressions for $h'(x)$ and $h''(x)$ and simplify where appropriate.

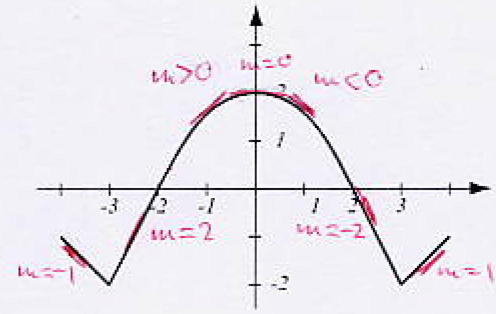
$h'(x) = f'(x)\sin x + f(x)\cos x$

$h''(x) = f''(x)\sin x + f'(x)\cos x + f'(x)\cos x + f(x)(-\sin x)$

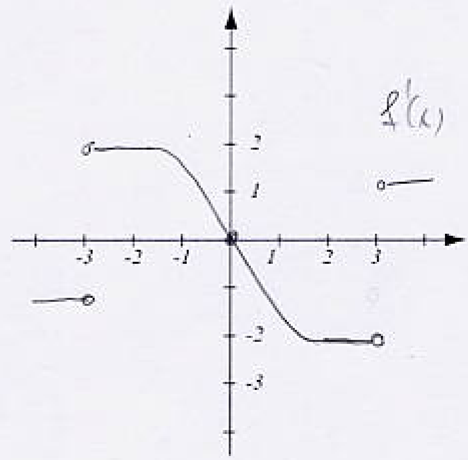
$= f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$

7. (12pts) The graph of the function $f(x)$ is shown at right.

a) Where is $f(x)$ not differentiable?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?



a) Not diff at $x = -3, 3$
(sharp points)



c) f is even
f' is odd
(true in general: If f is even, f' is odd)

8. (16pts) Let $f(x) = \dfrac{1}{3x+2}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(3, \frac{1}{11})$.

a) $$f'(x) = \lim_{h\to 0} \frac{\frac{1}{3(x+h)+2} - \frac{1}{3x+2}}{h} = \lim_{h\to 0} \frac{3x+2-(3(x+h)+2)}{(3(x+h)+2)(3x+2)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{3x+2-(3x+3h+2)}{(3x+3h+2)(3x+2)\cdot h} = \lim_{h\to 0} \frac{-3\cancel{h}}{(3x+3h+2)(3x+2)\cancel{h}} = \frac{-3}{(3x+0+2)(3x+2)}$$

$$= -\frac{3}{(3x+2)^2}$$

b) $f(x) = (3x+2)^{-1}$
$$f'(x) = -(3x+2)^{-2} \cdot 3$$
$$= -\frac{3}{(3x+2)^2}$$

c) $f'(3) = -\frac{3}{(9+2)^2} = -\frac{3}{121}$

$$y - \frac{1}{11} = -\frac{3}{121}(x-3)$$

$$y = -\frac{3}{121}x + \frac{9}{121} + \frac{1}{11} = -\frac{3}{121}x + \frac{20}{121}$$

9. (16pts) A first-generation iPhone is thrown upwards from ground level with initial velocity 40m/s (what else to do with a first-generation device!). Its position is given by the formula $s(t) = -5t^2 + 40t$.
a) Write the formula for the velocity of the iPhone at time t.
b) When does the iPhone reach height 60 meters?
c) What is the velocity of the iPhone when it reaches height 60 meters on its way up? On its way down?
d) What is the height of the iPhone when its velocity is 10m/s?

a) $v(t) = s'(t) = -10t + 40$

b) $-5t^2 + 40t = 60$

$5t^2 - 40t + 60 = 0 \quad | \div 5$

$t^2 - 8t + 12 = 0$

$(t-6)(t-2) = 0$

$t = 2, 6$

c) $v(2) = 20$ m/s

$v(6) = -20$ m/s

d) $-10t + 40 = 10$

$-10t = -30$

$t = 3$ s

$s(3) = -45 + 120 = 75$ m

10. (6pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Once you've found f and a, find the limit — it is equal to $f'(a)$!

$$\lim_{x \to \frac{\pi}{3}} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$a = \frac{\pi}{3}$

$f(x) = \cos x$

$f\left(\frac{\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$

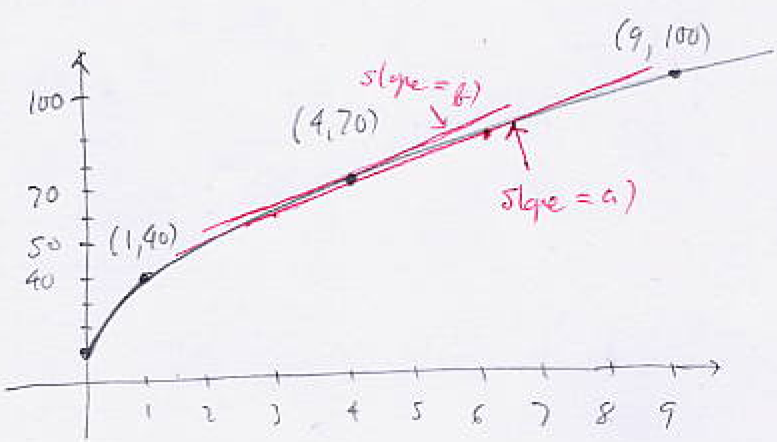
$f'(x) = -\sin x$

so $f'\left(\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

11. (12pts) The temperature (in degrees Celsius) of water being heated in a pan is given by $f(t) = 10 + 30\sqrt{t}$, where t is in minutes.
a) What is the average rate of warming from $t = 4$ to $t = 6$? What are the units?
b) What is the instantaneous rate of warming when $t = 4$? What are the units?
c) Draw the graph of f and state the geometric interpretation of the numbers you got above.

a) $\dfrac{f(6)-f(4)}{6-4} = \dfrac{10+30\sqrt{6}-(10+30\sqrt{4})}{6-4} = \dfrac{30(\sqrt{6}-2)}{2} = 15(\sqrt{6}-2)$ °C/min

b) $f'(t) = \dfrac{30}{2\sqrt{t}} = \dfrac{15}{\sqrt{t}}$ $\quad f'(4) = \dfrac{15}{2} = 7{,}5$ °C/min



a) is slope of secant line through (4,70) and $(6, 30\sqrt{6}+10)$

b) is slope of tangent line at (4,70)

Bonus. (10pts) Take the derivative of the function below and simplify. Go to town!

$$\frac{d}{dx}\sqrt{\frac{x+\sqrt{x}}{x-\sqrt{x}}} = \frac{1}{2\sqrt{\frac{x+\sqrt{x}}{x-\sqrt{x}}}} \cdot \frac{\left(1+\frac{1}{2\sqrt{x}}\right)(x-\sqrt{x}) - (x+\sqrt{x})\left(1-\frac{1}{2\sqrt{x}}\right)}{(x-\sqrt{x})^2}$$

$$\frac{x}{2\sqrt{x}} = \frac{\sqrt{x}\sqrt{x}}{2\sqrt{x}} = \frac{\sqrt{x}}{2}$$

$$= \frac{\sqrt{x-\sqrt{x}}}{2\sqrt{x+\sqrt{x}}} \cdot \frac{\left(x+\frac{\sqrt{x}}{2}-\sqrt{x}-\frac{1}{2}\right) - \left(x+\sqrt{x}-\frac{\sqrt{x}}{2}-\frac{1}{2}\right)}{(x-\sqrt{x})^2}$$

$$= \frac{\sqrt{x-\sqrt{x}}}{2\sqrt{x+\sqrt{x}}} \cdot \frac{\cancel{x}+\frac{\sqrt{x}}{2}-\sqrt{x}-\cancel{\frac{1}{2}}-\cancel{x}-\sqrt{x}+\frac{\sqrt{x}}{2}+\cancel{\frac{1}{2}}}{(x-\sqrt{x})^2} = \frac{-\sqrt{x}}{2\sqrt{x+\sqrt{x}}\,\underbrace{(x-\sqrt{x})^{3/2}}_{\sqrt{x-\sqrt{x}}\,(x-\sqrt{x})}} = \frac{-\sqrt{x}}{2\sqrt{x^2-x}\,(x-\sqrt{x})}$$