Introduction to Topology — Exam 1
MAT 516/616 , Fall 2013 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let X be a set. Define what a topology on X is.

Theory 2. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define $\operatorname{Int} A$.

Theory 3. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. State the theorem that reveals the relationship between $\operatorname{Int} A$, $\operatorname{Bd} A$ and $\operatorname{Ext} A$.

Type A problems (5pts each)

A1. Let $X = \{a, b, c\}$. Which of the following collections is a topology on X?

$\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{T}_3 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

A2. Let $A = (-2, 2) \cup \{5\}$ be a subset of the topological space $(\mathbf{R}, \mathcal{H})$. Find A', $\operatorname{Int} A$ and $\operatorname{Bd} A$.

A3. Let $A = \mathbf{Q} \cap [0, 1]$ (all rationals between 0 and 1) be a subset of the topological space $(\mathbf{R}, \mathcal{U})$. Determine $\operatorname{Cl} A$.

A4. Let A be a subset of a the topological space $(X, \mathcal{T})$. Show that $\operatorname{Cl} A = \operatorname{Int} A$ if and only if A is both open and closed. (Hint: don't do anything complicated. Use properties of interior and closure.)

A5. Let $(X, \mathcal{T})$ be a topological space. Show that a subset $A \subseteq X$ is dense if and only if for every open set U, $U \cap A \neq \emptyset$.

A6. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be the function between topological spaces defined by $f(x) = y_0$, where y_0 is a fixed element of Y. Show that f is continuous.

A7. Let X be any set with three elements or more and $\mathcal{B}$ be the collection of all two-element subsets of X. Show that $\mathcal{B}$ is not a base for any topology.

Type B problems (8pts each)

B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous   b) $\mathcal{C}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x - 1, & \text{if } x < 0. \end{cases}$$

B2. Show that the linear function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = mx + b$, $m > 0$ is $\mathcal{C}$–$\mathcal{C}$ continuous. How about when $m = 0$ or $m < 0$?

B3. For every subset A of a topological space X show that $\operatorname{Cl} A = A \cup \operatorname{Bd} A$.

B4. Show that the collection $\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in \mathbf{R}, n \in \mathbf{N}\}$ is a base for $(\mathbf{R},\mathcal{U})$, where the balls are defined using the usual metric $d(x, y) = |x - y|$.

B5. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a function between topological spaces and let $\mathcal{B}$ be a base for $\mathcal{S}$. Show that f is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

B6. Show that the function $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + 5|y_1 - y_2|$ is a metric on $\mathbf{R}^2$. Determine what $B_1(0, 0)$ is.

B7. Show that the function d defined below is a metric on $\mathbf{N}$ (natural numbers).

$$d(m, n) = \begin{cases} 0, & \text{if } m = n \\ \frac{1}{2}, & \text{if } m \neq n \text{ and both are even or both are odd} \\ 1 & \text{if one is even and the other is odd.} \end{cases}$$

Type C problems (12pts each)

C1. A collection $\mathcal{T}$ of subsets of the set of natural numbers $\mathbf{N}$ is defined as follows: $U \in \mathcal{T}$ provided that for every $m \in U$, all the divisors of m are also in U, where 1 is considered a divisor.

For example: $\{1, 2, 3, 4, 6, 12\}$, $\{1, 2, 4, 8\}$, $\{1, 7\}$, $\{3^k \mid k \geq 0\}$ are in $\mathcal{T}$;
$\{2, 5, 10\}$, $\{14, 28\}$, $\{1, 5, 6, 30\}$ are not in $\mathcal{T}$.

a) Show that $\mathcal{T}$ is a topology on $\mathbf{N}$.
b) Find $\operatorname{Cl} A$, where $A = \{15\}$.

C2. For any set A in a topological space X, show that $\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int} A))) = \operatorname{Cl}(\operatorname{Int} A)$. Show also that $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} A))) = \operatorname{Int}(\operatorname{Cl} A)$. (Hint: don't do anything complicated. Use properties of interior and closure.)

Introduction to Topology — Exam 2 MAT 516/616 , Fall 2013 — D. Ivanšić

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Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let X be a topological space and $A \subseteq X$. Define the relative topology on A.

Theory 2. (3pts) Define the neighborhood of point.

Theory 3. (3pts) Let X, $Y_1, \dots, Y_n$ be topological spaces. State the theorem that gives a criterion for when a function $f : X \to Y_1 \times \cdots \times Y_n$ is continuous, where $Y_1 \times \cdots \times Y_n$ has the product topology.

Type A problems (5pts each)

A1. Let $A = (-\infty, 7]$ be a subspace of $(\mathbf{R}, \mathcal{U})$. Which of the subsets of A are open in the relative topology: $(-1, 3)$, $(-\infty, 4]$, $(0, 7]$? Prove your answers.

A2. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. Let $A = \{a, b, c\}$. Find $\operatorname{Int}\{a, c\}$ and $\operatorname{Int}_A\{a, c\}$ and justify your answer.

A3. Let X be a topological space and let $A \subset X$ be a closed set. Show that $F \subseteq A$ is closed in A if and only if F is closed in X.

A4. Give an example of a function $f : (\mathbf{R}, \mathcal{H}) \to (\mathbf{R}, \mathcal{H})$ that is not continuous. Then give an example of a function that is not open.

A5. Let X be a topological space and define $f : X \times X \to X \times X$ as $f(x, y) = (y, x)$. Show that f is a homeomorphism. What is its inverse?

A6. Group the subspaces of $\mathbf{R}^2$ into groups of homeomorphic spaces. Show spaces from one pair of groups are not homeomorphic.

Type B problems (8pts each)

B1. Let X and Y be topological spaces with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Show that the collection $\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a base for the product space $X \times Y$.

B2. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}\}$. Find all the homeomorphisms $f : X \to X$ and justify that your list is exhaustive.

B3. Let X have the discrete topology. Show that the product topology on $X \times X$ is the discrete topology (i.e. every subset of $X \times X$ is open in the product topology).

B4. Let X be a topological space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x, x) \in X \times X \mid x \in X\}$.
a) Draw the set Z for the closed interval $X = [1, 2]$.
b) Show that X is homeomorphic to Z (Z has the relative topology).

B5. Let $\mathbf{R}$ have the topology $\mathcal{C}$. Show with a picture that the multiplication function $m : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is not continuous. (Hint: what are basic open sets in $\mathbf{R} \times \mathbf{R}$ like?)

Type C problems (12pts each)

C1. Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. Show that

$$\operatorname{Bd}(A \times B) = (\operatorname{Bd} A \times \operatorname{Cl} B) \cup (\operatorname{Cl} A \times \operatorname{Bd} B).$$

Illustrate for $X, Y = \mathbf{R}$, $A = (2, 4)$, $B = (1, 3)$. (Hint: you will not need to go into definitions, just use established properties.)

C2. Let $X = \mathbf{R}^2 - \{(0, 0)\}$ and $Y = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 > 1\}$. Show that X and Y are homeomorphic. First sketch what the homeomorphism does, then write a formula for it and prove it is a homeomorphism.

Introduction to Topology — Exam 3 MAT 516/616 , Fall 2013 — D. Ivanšić

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Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a topological space X is connected.

Theory 2. (3pts) Define a Hausdorff space.

Theory 3. (3pts) State the theorem that characterizes compact subsets of $\mathbf{R}$ (Heine-Borel Theorem).

Type A problems (5pts each)

A1. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, and let $A = \{a, b\}$. Is X connected? Is A connected? Prove your answers.

A2. Show that any subset of $(\mathbf{R}, \mathcal{H})$ with at least two elements is disconnected.

A3. Let $\mathbf{R}$ have the topology $\mathcal{T} = \{U \subseteq \mathbf{R} \mid U = \mathbf{R} \text{ or } [-2, 2] \subseteq U^c\}$. Is $(\mathbf{R}, \mathcal{T})$ connected?

A4. Is the set $\mathbf{Q} \cap [0, 1]$ (with usual topology) compact? Prove your answer.

A5. Use the Intermediate Value Theorem to show there exists a real number c satisfying $c^3 = 5$.

A6. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$.

A7. Show that $[3, \infty)$ is a compact subset of $(\mathbf{R}, \mathcal{C})$.

Type B problems (8pts each)

B1. Show that any connected subset A of $(\mathbf{R}, \mathcal{U})$ has the property: if $x < y$ are elements of A, then any $z \in (x, y)$ is also in A.

B2. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be the set $A = ([0, 1) \times \mathbf{R}) \cup \{(1, n) \mid n \in \mathbf{Z}\}$. Is A connected? Prove your answer.

B3. Let X be a topological space and A a finite subset of X. Show that A is compact.

B4. Show that the punctured plane $\mathbf{R} \times \mathbf{R} - \{(0, 0)\}$ is a connected space, with the topology relative to the usual product topology on $\mathbf{R} \times \mathbf{R}$. (Use known theorems and properties of connectedness rather than the definition.)

B5. Let X be a Hausdorff space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x, x) \in X \times X \mid x \in X\}$. Show that Z is a closed subset of $X \times X$.

Type C problems (12pts each)

C1. Let X be the square $[0,1] \times [0,1]$ with the usual product topology and $f : X \to \mathbf{R}$ a continuous function so that $f(0,0) = -3$ and $f(1,1) = 2$. Show that there is a point (c,d) in $(0,1) \times (0,1)$ (the "interior") such that $f(c,d) = 0$.

C2. Give an example of a compact subset of $(\mathbf{R}, \mathcal{U})$ that is not a closed interval. Then show: if $A \subseteq (\mathbf{R}, \mathcal{U})$ is a compact and connected subset, then A is a closed interval.

Introduction to Topology — Final Exam
MAT 516/616 , Fall 2013 — D. Ivanšić

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Theory 1. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define $\operatorname{Cl} A$.

Theory 2. (3pts) Define the product topology.

Theory 3. (3pts) Define when a topological space X is compact.

Type A problems (5pts each)

A1. Let $(X, \mathcal{T})$ be a topological space. Show that a subset $A \subseteq X$ is dense if and only if for every open set U, $U \cap A \neq \emptyset$.

A2. Let X be any set with three elements or more and $\mathcal{B}$ be the collection of all two-element subsets of X. Show that $\mathcal{B}$ is not a base for any topology.

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A6. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$.

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B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous b) $\mathcal{C}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x > 0 \\ x - 1, & \text{if } x \leq 0. \end{cases}$$

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