Fall '05: MAT 516, Exam 1

Do five problems, not all of which are the same type.
If you do more than five, best five will be counted.

Type 1 problems (5pts each)

A1. Let $A_\alpha \subset X$, for every $\alpha \in I$, and let $B \subset X$. Show that $B \subset \cup_{\alpha \in I} A_\alpha$ if and only if $B \subset A_\alpha$ for every $\alpha \in I$.

A2. Let $f : X \to Y$ be a one-to-one function. Show that for every subset $A \subset X$, $f^{-1}(f(A)) = A$. Give an example of a non-one-to-one function where equality does not hold.

A3. Let A be any set. Devise a one-to-one function $A \to 2^A$ and justify.

A4. Let $(\mathbf{R}^2, d)$ be a metric spaces, where

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, d_{\text{discrete}}(y_1, y_2)\}.$$

Describe the balls in this space.

A5. Let X be the set of continuous functions $f : [a, b] \to \mathbf{R}$. Show that the function d defined as $d(f, g) = \int_a^b |f(t) - g(t)|\, dt$ is a metric on X.

Type 2 problems (8pts each)

B1. For every real number $\alpha \in (1, 2)$ define a subset $A_\alpha \subset \mathbf{R}$ by $A_\alpha = (-\alpha/2, \alpha/2) \cup \{\alpha\}$ (so, an open interval and a point). Determine what $\cup_{\alpha \in I} A_\alpha$ and $\cap_{\alpha \in I} A_\alpha$ are and justify.

B2. Define a relation on the set of real numbers: $x \sim y$ if $x - y \in \mathbf{Q}$. Show that this is an equivalence relation.

B3. Show that the function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2$ is continuous.

B4. The function d_p is a metric on $\mathbf{R}^n$, where $d_p(x, y) = \sqrt[p]{\sum_{i=1}^n |x_i - y_i|^p}$. Prove the following inequalities:

$$d_\infty(x, y) \leq d_p(x, y) \leq \sqrt[p]{n}\, d_\infty(x, y).$$

B5. Let $\mathbf{R}^2$ have the standard straight-line distance function d_2. Draw an example (with justification) of:
a) an open set in $\mathbf{R}^2$.
b) a closed set in $\mathbf{R}^2$.
c) of a set in $\mathbf{R}^2$ that is neither open or closed.

B6. Let (X, d_{discrete}) be any set with the discrete metric. For any subset $A \subset X$, show that $A' = \emptyset$. (A' is the set of all limit points.)

Type 3 problems (12pts each)

C1. Let A be a set. Show that no function $f : A \to 2^A$ is onto. (Hint: assume the existence of an onto function $f : A \to 2^A$ and consider this subset of A: $C = \{x \mid x \notin f(x)\}$.)

C2. Let X be the set of all functions $f : [a, b] \to \mathbf{R}$ that have a continuous derivative, and let Y be the set of all continuous functions $f : [a, b] \to \mathbf{R}$. Put the metric

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$$

on both X and Y. Show that the derivative function $D : (X, d) \to (Y, d)$, $D(f) = f'$ is *not* continuous. Hint: Theorem 5.4.

<u>Fall ’05: MAT 516, Exam 2</u>

Do five problems, not all of which are the same type.
If you do more than five, best five will be counted.

Type 1 problems (5pts each)

A1. Show that the topology $(X, 2^X)$ is metrizable.

A2. Let $A = \{\frac{1}{n} \mid n \in \mathbf{N}\}$ be a subset of $\mathbf{R}$, where $\mathbf{R}$ has the standard topology. Find $\overline{A}$, $\operatorname{Int} A$ and $\operatorname{Bd} A$ and justify with pictures.

A3. Consider the sets $\mathbf{N}$ and $2\mathbf{N}$ (set of even naturals) and put the finite-complement topology on both of them (U is open if it is the complement of a finite set or if $U = \emptyset$.) Show that $(\mathbf{N}, \mathcal{T})$ and $(2\mathbf{N}, \mathcal{T}')$ are homeomorphic.

A4. Find an example of a discontinuous function $f : \mathbf{R} \to \mathbf{R}$ and a set $A \subset \mathbf{R}$ for which $f(\overline{A}) \not\subset \overline{f(A)}$.

A5. Find an example of a space X and subsets A, U where $U \subset A$ is open in A but not open in X. Similarly, find an example of a closed set $F \subset A$ that is closed in A but not closed in X. (A has the subspace topology, the space X and set A in your two examples need not be the same.)

Type 2 problems (8pts each)

B1. Let $\mathcal{T}_{(-a,a)} = \{(-a, a) \subset \mathbf{R} \mid a \in \mathbf{R}\} \cup \{\mathbf{R}, \emptyset\}$ be a collection of subsets of $\mathbf{R}$ (it consists of open intervals symmetric about 0). Show $\mathcal{T}_{(-a,a)}$ is a topology on $\mathbf{R}$.

B2. For every subset A of a topological space X show that $\overline{A} = A \cup \operatorname{Bd} A$.

B3. Let (X, d) be a metric space. Show that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

B4. Let $(\mathbf{R}^2, d)$ be a metric space, where $d((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, d_{\text{discrete}}(y_1, y_2)\}$. Let $A = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\} \cup \{x\text{-axis}\}$. Find $\overline{A}$, $\operatorname{Int} A$ and $\operatorname{Bd} A$ and justify with pictures.

B5. Let $a, b, c, d \in \mathbf{R}$. Show that the intervals (a, b) and (c, d) are homeomorphic. The topology on both intervals is the subspace topology. Justify carefully why your proposed homeomorphism is continuous.

B6. If X and Y are topological spaces, show that $X \times Y$ is homeomorphic to $Y \times X$ (both have product topology).

B7. Show that the collection $\mathcal{B} = \{(a, b) \subset \mathbf{R} \mid a, b \in \mathbf{Q}\}$ (open intervals with rational endpoints) is a basis and that the topology generated by it is the same as the standard topology.

B8. Show that the set $\mathbf{Q} \times \mathbf{Q}$ is dense in $\mathbf{R} \times \mathbf{R}$ (standard topology on $\mathbf{R}$, product topology on $\mathbf{R} \times \mathbf{R}$).

Type 3 problems (12pts each)

C1. Let $X_1, \ldots X_n$ be topological spaces and equip $X_1 \times \cdots \times X_n$ with the product topology. If Y is a topological space, show that $f : Y \to X_1 \times \cdots \times X_n$ is continuous if and only if $f_i = p_i f : Y \to X_i$ is continuous for every $i = 1, \ldots, n$ (p_i is the projection onto the i-th coordinate).

C2. Let X and Y be topological spaces and choose $b \in Y$. Let $i_b : X \to X \times Y$ be defined by $i_b(x) = (x, b)$. Show that i_b is a homeomorphism from X to $X \times \{b\}$, where $X \times \{b\} \subset X \times Y$ has the subspace topology inherited from the product topology on $X \times Y$.

C3. For any set A in a topological space X, show that $\overline{\operatorname{Int}\left(\overline{\operatorname{Int} A}\right)} = \overline{\operatorname{Int} A}$. Show also that $\operatorname{Int}\left(\overline{\operatorname{Int}\,(\overline{A})}\right) = \operatorname{Int}(\overline{A})$. (Hint: don't do anything complicated. Use properties of interior and closure.)

<u>Fall '05: MAT 516, Exam 3</u>

Do five problems, not all of which are the same type.
If you do more than five, best five will be counted.
You may quote theorems from book or lectures on solutions, but
you may not quote homework problems as all or part of a solution.

Type 1 problems (5pts each)

A1. Consider the set $\mathbf{N}$ with the finite-complement topology. Show that $\mathbf{N}$ is connected.

A2. Let A be a connected subset of a topological space X, and let U be an open and closed set in X. Show that either $U \cap A = \emptyset$ or $A \subset U$.

A3. Let X be any connected topological space, $f : X \to \mathbf{R}$ a continuous function. Suppose there exist $a, b \in X$ so that $f(a) \neq f(b)$. Show that for every real number N between $f(a)$ and $f(b)$ there exists a $c \in X$ so that $f(c) = N$.

A4. Use theorems from the book to justify that $\mathbf{R} \times \mathbf{R}$ with the product topology is connected.

A5. Let $f : X \to Y$ be a continuous function between topological spaces X and Y. Show that $f(\text{cmp}(a)) \subset \text{cmp}(f(a))$ for every $a \in X$.

A6. Let $f : X \to Y$ be a homeomorphism between topological spaces X and Y and suppose $f(x_0) = y_0$ for some $x_0 \in X$, $y_0 \in Y$. Show that the restriction of f is a homeomorphism from $X - \{x_0\}$ to $Y - \{y_0\}$.

Type 2 problems (8pts each)

B1. In a topological space X, let there exist subsets A and B such that $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset$. Show that $A \cup B$ is disconnected.

B2. Let $X = \mathbf{R}$ have the lower limit topology (basis for this topology is all intervals of form $[a, b)$, $a < b$). Identify the components of X.

B3. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show there exists a number $c \in [0, 1]$ so that $f(c) = c^2$.

B4. Let $f : \mathbf{R} \to \mathbf{N}$ be a continuous function, where $\mathbf{N}$ has the discrete topology. Show that f is constant.

B5. Show that $[0, 1]$ is not homeomorphic to the circle S^1. (Hint: problem **A6**, which you need not do to use here, and connectedness.)

Type 3 problems (12pts each)

C1. Let U be a connected open subset of $\mathbf{R}^2$, which has the standard topology, and let $x_0 \in U$.
a) Show that the set $\{x \in U \mid x \text{ can be joined to } x_0 \text{ by a path in } U\}$ is open.
b) Show that the set $\{x \in U \mid x \text{ cannot be joined to } x_0 \text{ by a path in } U\}$ is open.
c) Deduce that U is path-conected.

C2. Let X be the set of all continuous functions $f : [a, b] \to \mathbf{R}$ with the metric

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|.$$

Show that X is path connected. Hint: if $f, g \in X$, the path connecting them is $H(t) = (1 - t)f + tg$. The task here is to show this is a continuous function $[0, 1] \to X$. Use an ε, δ argument.