Introduction to Topology — Final List **MAT 516/616 , Fall 2013 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1, 2 & 3. (3pts each) Surprise! These can come from the whole course, and will not be the ones I already asked.

Note: The actual final exam will not include all of the following problems. Therefore one cannot just learn the solutions to five from this selection and do those on the exam.

Type A problems (5pts each)

A1. Let A be a subset of a the topological space $(X, \mathcal{T})$. Show that $\operatorname{Cl} A = \operatorname{Int} A$ if and only if A is both open and closed. (Hint: don't do anything complicated. Use properties of interior and closure.)

A2. Let $(X, \mathcal{T})$ be a topological space. Show that a subset $A \subseteq X$ is dense if and only if for every open set U, $U \cap A \neq \emptyset$.

A3. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be the function between topological spaces defined by $f(x) = y_0$, where y_0 is a fixed element of Y. Show that f is continuous.

A4. Let X be any set with three elements or more and $\mathcal{B}$ be the collection of all two-element subsets of X. Show that $\mathcal{B}$ is not a base for any topology.

A5. Let X be a topological space and let $A \subset X$ be a closed set. Show that $F \subseteq A$ is closed in A if and only if F is closed in X.

A6. Let X be a topological space and define $f : X \times X \to X \times X$ as $f(x, y) = (y, x)$. Show that f is a homeomorphism. What is its inverse?

A7. Show that any subset of $(\mathbf{R}, \mathcal{H})$ with at least two elements is disconnected.

A8. Let $\mathbf{R}$ have the topology $\mathcal{T} = \{U \subseteq \mathbf{R} \mid U = \mathbf{R} \text{ or } [-2, 2] \subseteq U^c\}$. Is $(\mathbf{R}, \mathcal{T})$ connected?

A9. Is the set $\mathbf{Q} \cap [0, 1]$ (with usual topology) compact? Prove your answer.

A10. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$.

A11. Show that $[3, \infty)$ is a compact subset of $(\mathbf{R}, \mathcal{C})$.

Type B problems (8pts each)

Note that **B1** has been changed from its original (mistaken) form.

B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous b) $\mathcal{C}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x > 0 \\ x - 1, & \text{if } x \leq 0. \end{cases}$$

B2. Show that the collection $\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in \mathbf{R}, n \in \mathbf{N}\}$ is a base for $(\mathbf{R}, \mathcal{U})$, where the balls are defined using the usual metric $d(x, y) = |x - y|$.

B3. Let X have the discrete topology. Show that the product topology on $X \times X$ is the discrete topology (i.e. every subset of $X \times X$ is open in the product topology).

B4. Show that any connected subset A of $(\mathbf{R}, \mathcal{U})$ has the property: if $x < y$ are elements of A, then any $z \in (x, y)$ is also in A.

B5. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be the set $A = ([0, 1) \times \mathbf{R}) \cup \{(1, n) \mid n \in \mathbf{Z}\}$. Is A connected? Prove your answer.

B6. Show that the punctured plane $\mathbf{R} \times \mathbf{R} - \{(0, 0)\}$ is a connected space, with the topology relative to the usual product topology on $\mathbf{R} \times \mathbf{R}$. (Use known theorems and properties of connectedness rather than the definition.)

B7. Let X be a Hausdorff space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x, x) \in X \times X \mid x \in X\}$. Show that Z is a closed subset of $X \times X$.

Type C problems (12pts each)

C1. A collection $\mathcal{T}$ of subsets of the set of natural numbers $\mathbf{N}$ is defined as follows: $U \in \mathcal{T}$ provided that for every $m \in U$, all the divisors of m are also in U, where 1 is considered a divisor.

For example: $\{1,2,3,4,6,12\}$, $\{1,2,4,8\}$, $\{1,7\}$, $\{3^k \mid k \geq 0\}$ are in $\mathcal{T}$;
$\{2,5,10\}$, $\{14,28\}$, $\{1,5,6,30\}$ are not in $\mathcal{T}$.

a) Show that $\mathcal{T}$ is a topology on $\mathbf{N}$.
b) Find $\operatorname{Cl} A$, where $A = \{15\}$.

C2. Let $X = \mathbf{R}^2 - \{(0,0)\}$ and $Y = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 > 1\}$. Show that X and Y are homeomorphic. First sketch what the homeomorphism does, then write a formula for it and prove it is a homeomorphism.

C3. Let X be the square $[0,1] \times [0,1]$ with the usual product topology and $f : X \to \mathbf{R}$ a continuous function so that $f(0,0) = -3$ and $f(1,1) = 2$. Show that there is a point (c,d) in $(0,1) \times (0,1)$ (the "interior") such that $f(c,d) = 0$.

C4. Give an example of a compact subset of $(\mathbf{R}, \mathcal{U})$ that is not a closed interval. Then show: if $A \subseteq (\mathbf{R}, \mathcal{U})$ is a compact and connected subset, then A is a closed interval.