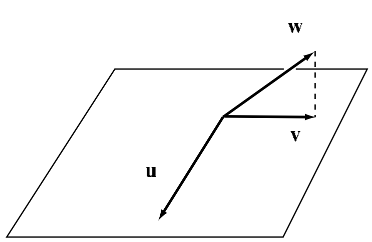
Calculus 3 — Exam 1
MAT 309, Fall 2013 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (16pts) Let $\mathbf{u} = \langle 1, 1, -4 \rangle$ and $\mathbf{v} = \langle 3, -1, 0 \rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - 4\mathbf{v}$, and $|\mathbf{u}|$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. (12pts) In the picture, vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and the projection of vector $\mathbf{w}$ to the vector $\mathbf{v}$ is the vector $\mathbf{v}$. The angle between vectors $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. Suppose that $|\mathbf{u}| = 7$, $|\mathbf{v}| = 3$ and $|\mathbf{w}| = 2\sqrt{3}$. Draw the vector $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ in the picture. What is its length?



3. (8pts) Draw the region in $\mathbf{R}^3$ described by:

$$\frac{x^2}{9} + \frac{z^2}{4} = 1, \; x \geq 0$$

4. (6pts) Which of the following expressions are meaningful? Briefly explain.

$\mathbf{u} \times (\mathbf{v} \cdot \mathbf{u})$ $\quad$ $(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ $\quad$ $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v})$

5. (20pts) A parallelogram in $\mathbf{R}^3$ has vertices $A = (3, 1, 4)$, $B = (5, 2, 4)$, $C = (8, 1, 3)$, and $D = (6, 0, 3)$.
a) Find the equation of the plane that contains this parallelogram.
b) Find the area of the parallelogram.
c) Is this parallelogram a rectangle?

6. (22pts) The curve $\mathbf{r}(t) = \langle (t+1)\cos t, (t+1)\sin t, 2t \rangle$ is given, $0 \le t \le 4\pi$.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = 2\pi$ and sketch the tangent line.
c) Set up the integral for the length of the curve. Simplify the function inside the integral as much as possible, but do not evaluate the integral.

7. (16pts) This problem is about the surface $\frac{x^2}{4} - \frac{y^2}{25} - \frac{z^2}{16} = 1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

Bonus (10pts) A ray of light, represented by the line $x = 2 - t$, $y = 4 + 2t$, $z = -3 - 3t$ reflects off the mirror represented by the plane $x - y + 2z = 10$ at point $P = (4, 0, 3)$. Find parametric equations of the line that represents the reflected ray. *(Hints: the ray and the reflected ray determine a plane that is perpendicular to the mirror. Vector projection is helpful.)*

Calculus 3 — Exam 2 **MAT 309, Fall 2013 — D. Ivanšić**

Name:________________________________

Show all your work!

1. (22pts) Let $T(x,y) = \dfrac{y}{x^2}$.
a) Find the domain of T.
b) Sketch the contour map for the function, drawing level curves for levels $k = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Note the domain on the picture.
c) Suppose T represents temperature in degrees Celsius in the plane, and a freezing bug located at $(2,-4)$ wishes to move to a point with a higher temperature. In what direction should it start moving to achieve the greatest increase in temperature? What is the directional derivative in that direction?
d) Draw a path the bug would take in order to reach a point with temperature 1°C if it always moves in the direction of the greatest increase of temperature.

2. (10pts) Find the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{25} + \dfrac{z^2}{16} = 1$ at the point $\left(\sqrt{2}, -\frac{5}{2}, -2\right)$. Simplify the equation to standard form.

3. (18pts) Let $B = \dfrac{x^2 + y^2}{x + 1}$, $x = \cos u + \sin v$, $y = \sin u \cos v$. Use the chain rule to find $\dfrac{\partial B}{\partial v}$ when $u = \frac{\pi}{4}$, $v = \pi$.

4. (16pts) The body surface area S in m^2 can be calculated from a person's weight w in kg and height h in cm using the formula $S = \dfrac{\sqrt{wh}}{60}$. Use differentials to estimate the change in body surface area of a woman who weighs 64kg and is 169cm tall if her weight decreases by 0.5kg and her height increases by 2cm. Substitute all the numbers, and simplify what you can, but stop when the numbers get hairy. (Note: $13^2 = 169$.)

5. (20pts) At a state fair, junked cars get catapulted at inital speed 25m/s and angle α for which $\tan\alpha = \frac{1}{2}$. Assume $g = 10$.

a) Find the position of the car at time t.

b) When does the car fall to the ground?

b) Find the horizontal distance that the car will travel.

6. (14pts) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ at the point $(1, 2, -1)$, if $yz^3 + xz^2 - x^2y^3 = -9$.

Bonus (10pts) Show that the bug in problem 1 moves along the ellipse $\frac{x^2}{36} + \frac{y^2}{18} = 1$. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla T(\mathbf{r}(t))$. *Hint: a parametrization to ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $x = a\cos t,\ y = b\sin t$.

Calculus 3 — Exam 3
MAT 309, Fall 2013 — D. Ivanšić

Name:______________________________

Show all your work!

1. (16pts) Find $\iint_D x\,dA$ if D is the region bounded by $y = 2 - 2x$ and $y = x^2 - 3x$. Sketch the region of integration first.

2. (18pts) Let D be the region bounded by the curves $x = 0$, $y = 2$ and $y = \sqrt{x}$. Sketch the region and set up $\iint_D (y^3 + 1)^5\,dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order.

3. (18pts) Use polar coordinates to find the area of the region that is inside the circle $(x-1)^2 + y^2 = 1$ and above the line $y = x$. Sketch the region of integration first.

4. (20pts) Find and classify the local extremes for $f(x, y) = x^3 + 3x^2y - y^3 + 9y$.

5. (12pts) Set up the triple iterated integral for $\iiint_E xe^{x+y+z}\,dV$, where E is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 5$. Sketch the region of integration first. Do not evaluate the integral.

6. (16pts) Sketch the region E that is in the first octant $(x, y, z \geq 0)$, and bounded by the plane $y = \frac{1}{2}x$ and the parabolic cylinder $z = 1 - y^2$. Then write the two iterated triple integrals that stand for $\iiint_W f\,dV$ which end in $dz\,dx\,dy$ and $dy\,dz\,dx$.

Bonus (10pts) Let $A = (0, 0)$, $B = (1, 0)$ and $C = (0, 2)$ and let d_A, d_B and d_C represent the distance from a point (x, y) to A, B and C, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2 + d_C^2$ among all points (x, y) in the triangle ABC (edges are included).

Calculus 3 — Exam 4
MAT 309, Fall 2013 — D. Ivanšić

Name: ____________________

Show all your work!

1. (18pts) Use cylindrical coordinates to find the volume of the region E enclosed by the paraboloids $z = x^2 + y^2$ and $z = 3 - \frac{1}{2}(x^2 + y^2)$. Sketch the region E.

2. (18pts) Use spherical coordinates to find $\iiint_E xz\,dV$, where E is the part of the first octant that is inside the sphere $x^2 + y^2 + z^2 = 16$, and outside the sphere $x^2 + y^2 + z^2 = 9$. Sketch the region E.

3. (14pts) Let $\mathbf{F}(x, y) = \langle x, 3 \rangle$.
a) Roughly draw the vector field $\mathbf{F}(x, y)$, scaling the vectors for a better picture.
b) Guess a function $f(x, y)$ so that $\mathbf{F} = \nabla f$.
c) How could you have roughly done a) without evaluating the vector field at various points?

4. (18pts) In both cases set up and simplify the set-up, but do not evaluate the integral.
a) $\int_C x^2 - y^2 + z^2 \, ds$, where C is the line segment from $(0, 0, 1)$ to $(1, -3, 3)$.
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x, y) = \langle xe^y, ye^x \rangle$, where C is the circle of radius 5 centered at the origin.

5. (12pts) Find the cylindrical and spherical coordinates of the point whose cartesian coordinates are $(-\sqrt{6}, -\sqrt{2}, 2\sqrt{2})$.

6. (20pts) Use change of variables to find $\iint_D y\, dA$, if D is the rectangle that is bounded by the lines $y = x$, $y = x + 5$, $y = -x$, $y = -x + 1$. Sketch the rectangle.

Bonus. (10pts) Find the Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\Phi)}$, where x, y, z are functions that convert spherical coordinates to cartesian. What do you expect to get?

Calculus 3 — Exam 5 MAT 309, Fall 2013 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (10pts) Let $f(x, y) = \dfrac{x^3}{y^2}$, and let $\mathbf{F} = \nabla f$. Apply the fundamental theorem for line integrals to answer:
a) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is part of the parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$? How about if C is a straight line segment from $(1, 1)$ to $(3, 9)$?
b) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the circle centered at $(3, 4)$ with radius 2?

2. (12pts) Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x, y, z) = \langle z^2 - 4y^2, 4x^2 - 3z^2, 3y^2 - x^2 \rangle$.

3. (14pts) A surface is parametrized by $\mathbf{r}(u,v) = \langle u^2, v^2, u+v\rangle$. Find the equation of the tangent plane to this surface at the point where $(u,v) = (2,-3)$.

4. (20pts) One of the two vectors fields below is not a gradient field, and the other one is (curl detects it). Identify which is which, and find the potential function for the one that is.

$\mathbf{F}(x,y,z) = \langle 2x\sin z + y^3e^x, 3y^2e^x + \cos z, x^2\cos z - y\sin z\rangle$ $\qquad$ $\mathbf{G}(x,y,z) = \langle x^2, y^2, yz^2\rangle$

5. (26pts) Consider the part of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$ between the planes $z = -2$ and $z = 0$.

a) Draw the surface, parametrize it and specify the planar region D where your parameters come from.

b) Set up the iterated integral that gives the area of the surface. Simplify the set-up, but do not evaluate the integral.

6. (18pts) Use Green's theorem to find the line integral $\int_C x^3\,dx + xy\,dy$, where C is the triangle from $(-1,0)$ to $(1,0)$ to $(0,1)$ to $(-1,0)$.

Bonus. (10pts) Use Green's theorem to find the area enclosed by the circle $x^2+y^2=4$ that is above the line $y=1$.

Calculus 3 — Final Exam **MAT 309, Fall 2013 — D. Ivanšić**

Name: ______________________________

Show all your work!

1. (12pts) Find the equation of the plane that contains the point $(3, 2, -4)$ and the line $x = 1 - 2t$, $y = 3 + t$, $z = -2 - t$.

2. (16pts) Let $f(x, y) = x^2 - y^2$.
a) Sketch the contour map for the function, drawing level curves for levels $k = -2, -1, 0, 1, 2$.
b) At point $(2, -1)$, what is the directional derivative of f in the direction of $\langle 1, -1 \rangle$?
c) In what direction is the directional derivative of f the greatest at $(2, -1)$, and what is it?
d) If C is the curve parametrized by $x = 2t - 1$, $y = t^2 + 3t$, $0 \le t \le 2$, what is $\int_C \nabla f \cdot d\mathbf{r}$?

3. (10pts) Find the equation of the tangent plane to the surface $\dfrac{x^2}{9} - \dfrac{y^2}{4} - \dfrac{z^2}{16} = 1$ at the point $(6, -2\sqrt{2}, 4)$. Simplify the equation to standard form.

4. (12pts) The volume of a cylinder is given by $V = \pi r^2 h$. When $r = 2$ meters and $h = 5$ meters, use differentials to estimate the change in volume of the cylinder, if its radius decreases by 0.1m and its height increases by 0.2m.

5. (16pts) Let D be the region bounded by the curves $x = 0$, $y = \frac{1}{3}$ and $y = e^x$. Sketch the region and set up $\iint_D y^2 \, dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order.

6. (18pts) Find and classify the local extremes for $f(x, y) = x^3 + 3x^2y - y^3 + 9y$.

7. (16pts) Use cylindrical or spherical coordinates to find $\iiint_E x^2 + y^2 \, dV$, where E is the region inside the sphere $x^2 + y^2 + z^2 = 9$ and above the cone $z = \sqrt{3(x^2 + y^2)}$ Sketch the region E.

Bonus (10pts) A ray of light, represented by the line $x = 2 - t$, $y = 4 + 2t$, $z = -3 - 3t$ reflects off the mirror represented by the plane $x - y + 2z = 10$ at point $P = (4, 0, 3)$. Find parametric equations of the line that represents the reflected ray. *(Hints: the ray and the reflected ray determine a plane that is perpendicular to the mirror. Vector projection is helpful.)*