Calculus 3 — Final Exam
MAT 309, Fall 2013 — D. Ivanšić

Name: Saul Ocean

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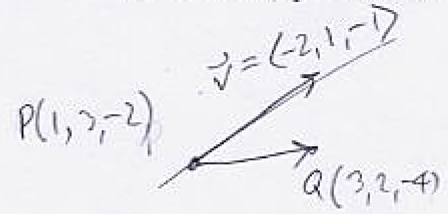
1. (12pts) Find the equation of the plane that contains the point $(3, 2, -4)$ and the line $x = 1 - 2t$, $y = 3 + t$, $z = -2 - t$.



$$\vec{n} = \vec{PQ} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -2 \\ -2 & 1 & -1 \end{vmatrix} = \langle 1-(-2), -(-2-4), 2-2 \rangle = \langle 3, 6, 0\rangle = 3\langle 1, 2, 0\rangle$$

$\vec{PQ} = \langle 2, -1, -2 \rangle$

Use $\vec{n} = \langle 1, 2, 0\rangle$

$$1\cdot(x-1) + 2(y-3) + 0(z-(-2)) = 0$$

$$x + 2y = 7$$

2. (16pts) Let $f(x, y) = x^2 - y^2$.

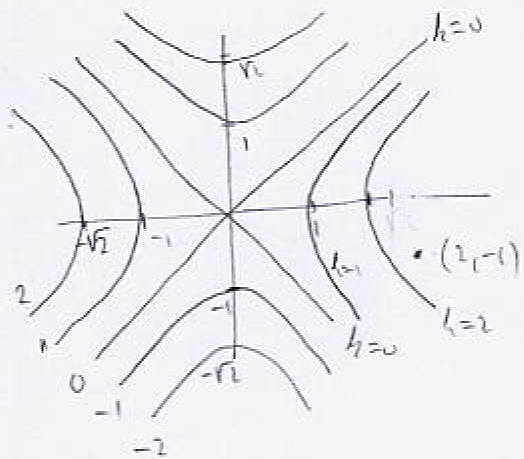
a) Sketch the contour map for the function, drawing level curves for levels $k = -2, -1, 0, 1, 2$.
b) At point $(2, -1)$, what is the directional derivative of f in the direction of $\langle 1, -1\rangle$?
c) In what direction is the directional derivative of f the greatest at $(2, -1)$, and what is it?
d) If C is the curve parametrized by $x = 2t - 1$, $y = t^2 + 3t$, $0 \le t \le 2$, what is $\int_C \nabla f \cdot d\mathbf{r}$?

a) $x^2 - y^2 = k$

Hyperbolas when $k \neq 0$

$x^2 - y^2 = 0$

$y = \pm x$ Two lines



b) $\nabla f = \langle 2x, -2y\rangle$

$\nabla f(2,-1) = \langle 4, 2\rangle$

$\vec{u} = \dfrac{\langle 1, -1\rangle}{\sqrt{1+1}} = \dfrac{1}{\sqrt{2}}\langle 1, -1\rangle$

$D_{\vec{u}} f(2,-1) = \langle 4, 2\rangle \cdot \dfrac{1}{\sqrt{2}}\langle 1, -1\rangle = \dfrac{1}{\sqrt{2}}(4-2) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

c) It is greatest in direction of $\nabla f = \langle 4, 2\rangle$ and equals $|\nabla f| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

d) $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0)) = f(3, 10) - f(-1, 0)$

$= (3^2 - 10^2) - ((-1)^2 - 0^2) = 9 - 100 - 1 = -92$

3. (10pts) Find the equation of the tangent plane to the surface $\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{16} = 1$ at the point $(6, -2\sqrt{2}, 4)$. Simplify the equation to standard form.

$f(x,y,z) = \frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{16}$

$\vec{n} = \nabla f = \left\langle \frac{2x}{9}, -\frac{2y}{4}, -\frac{2z}{16} \right\rangle = \left\langle \frac{2x}{9}, -\frac{y}{2}, -\frac{z}{8} \right\rangle$

$\nabla f(6, -2\sqrt{2}, 4) = \left\langle \frac{12}{9}, \frac{2\sqrt{2}}{2}, -\frac{4}{8} \right\rangle$

$= \left\langle \frac{4}{3}, \sqrt{2}, -\frac{1}{2} \right\rangle$

Use $\vec{n} = 6\nabla f = \langle 8, 6\sqrt{2}, -3 \rangle$

$8(x-6) + 6\sqrt{2}(y-(-2\sqrt{2})) - 3(z-4) = 0$

$8x + 6\sqrt{2}y - 3z - 48 + 24 + 12 = 0$

$8x + 6\sqrt{2}y - 3z = 12$

4. (12pts) The volume of a cylinder is given by $V = \pi r^2 h$. When $r = 2$ meters and $h = 5$ meters, use differentials to estimate the change in volume of the cylinder, if its radius decreases by 0.1m and its height increases by 0.2m.

$V(r,h) = \pi r^2 h$

$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$

$= \pi 2rh\, dr + \pi r^2 dh$

When $r = 2$ $\quad dr = -0.1$

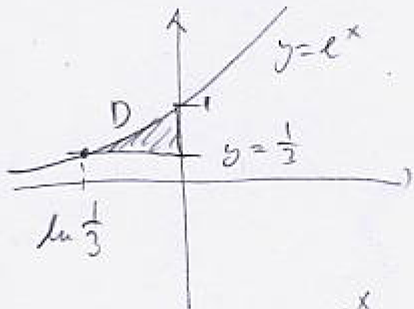
$h = 5 \quad dh = 0.2$

we get $\Delta V \approx dV = \pi\left(2 \cdot 2 \cdot 5 \cdot (-0.1) + 2^2 \cdot 0.2\right)$

$= \pi(-2 + 0.8) = \pi(-1.2) = -1.2\pi$

Volume decreases by about 1.2π units^3

5. (16pts) Let D be the region bounded by the curves $x = 0$, $y = \frac{1}{3}$ and $y = e^x$. Sketch the region and set up $\iint_D y^2\,dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order.



As type 1: $\displaystyle\int_{\ln\frac13}^{0}\int_{\frac13}^{e^x} y^2\,dy\,dx$

As type 2: $\displaystyle\int_{\frac13}^{1}\int_{\ln y}^{10} y^2\,dx\,dy$

$(e^{\ln\frac13})^3 = (\frac13)^3 = \frac{1}{27}$

$$\overset{=}{\uparrow}\int_{\ln\frac13}^{0} \frac{y^3}{3}\Big|_{1/3}^{e^x} dx = \int_{\ln\frac13}^{0} \frac13\left(e^{3x}-\left(\frac13\right)^3\right)dx = \frac{-1}{81}\left(0-\ln\frac13\right)+\frac13\left(\frac{e^{3x}}{3}\right)\Big|_{\ln\frac13}^{0}$$

w/ type 1

$$= -\frac{\ln 3}{81}+\frac19\left(1-\frac{1}{27}\right)$$

$$= -\frac{\ln 3}{81}+\frac19\,\frac{26}{27} = \frac{26-3\ln 3}{243}$$

6. (18pts) Find and classify the local extremes for $f(x,y) = x^3 + 3x^2y - y^3 + 9y$.

Critical pts:

$f_x = 3x^2+6xy$

$f_y = 3x^2-3y^2+9$

$\begin{cases} 3x^2+6xy=0 \\ 3x^2-3y^2+9=0 \end{cases}$

$3x(x+2y)=0$

$\Rightarrow x=0$ or $x=-2y$

$\Rightarrow -3y^2+9=0$ | $3(-2y)^2-3y^2+9=0$

$y^2=3$ | $9y^2+9=0$

$y=\pm\sqrt3$ | $y^2=-1$

no solution

$(0,\sqrt3)$, $(0,-\sqrt3)$

$D = \begin{vmatrix} 6x+6y & 6x \\ 6x & -6y \end{vmatrix}$

$D(0,\sqrt3) = \begin{vmatrix} 6\sqrt3 & 0 \\ 0 & -6\sqrt3 \end{vmatrix} < 0$

$D(0,-\sqrt3) = \begin{vmatrix} -6\sqrt3 & 0 \\ 0 & 6\sqrt3 \end{vmatrix} < 0$

Both are saddle points.

$$= 2\pi \int_0^{\frac32} r^3\sqrt{9-r^2} - \sqrt{3}\, r^4 \, dr = 2\pi\left(-\sqrt{3}\cdot\frac{r^5}{5}\Big|_0^{\frac32} + \int_9^{27/4} (9-u)\sqrt{u}\left(-\tfrac12 du\right)\right) = 2\pi\left(-\frac{243\sqrt{3}}{5\cdot 32} + \frac12\left(9\,\frac23 u^{\frac32} - \frac25 u^{\frac52}\right)\right)$$

$u = 9 - r^2$ $\quad r=\frac32, u=\frac{27}{4}$

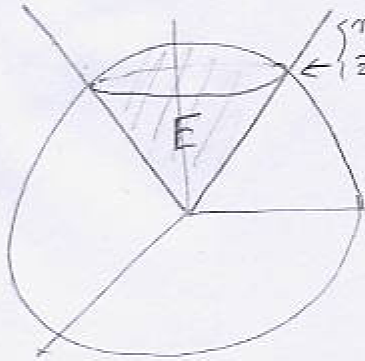
$du = -2r\,dr$ $\quad r=0, u=9$

$-\frac12 du = r\,dr$

$$= 2\pi\left(-\frac{243\sqrt{3}}{2} + 3\left(\left(\frac{27}{4}\right)^{\frac32} - 9^{\frac32}\right) - \frac15\left(\left(\frac{27}{4}\right)^{\frac52} - 9^{\frac52}\right)\right) \text{ etc.}$$

$9 - \frac94 = \frac{27}{4}$

7. (16pts) Use cylindrical or spherical coordinates to find $\iiint_E x^2 + y^2\, dV$, where E is the region inside the sphere $x^2+y^2+z^2 = 9$ and above the cone $z = \sqrt{3(x^2+y^2)}$ Sketch the region E.



$\begin{cases} r^2+z^2=9 \\ z=\sqrt{3}r \end{cases}$

$4r^2 = 9$

$r^2 = \frac94$

$r = \frac32$

Spherical:

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{3} \underbrace{r^2}_{\rho^2\sin^2\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{3} \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^3 \rho^4 d\rho \int_0^{\pi/6} \sin\phi\,(1-\cos^2\phi)\, d\phi = 2\pi\cdot\frac{\rho^5}{5}\Big|_0^3 \cdot \int_1^{\frac{\sqrt3}{2}} (1-u^2)(-du) = \int_{\sqrt3/2}^{1} (1-u^2)\,du$$

$u = \cos\phi$ $\quad \phi = \frac{\pi}{6}, u = \frac{\sqrt3}{2}$

$du = -\sin\phi\, d\phi$ $\quad \phi = 0, u = 1$

$$= 2\pi\cdot\frac{243}{5}\cdot\left(\left(1-\frac{\sqrt3}{2}\right) - \frac{u^3}{3}\Big|_{\sqrt3/2}^{1}\right) = \frac{2\pi\cdot 243}{5}\left(1-\frac{\sqrt3}{2} - \frac13\left(1-\frac{3\sqrt3}{8}\right)\right)$$

$$= \frac{2\pi\cdot 243}{5}\left(\frac23 + \frac{-9\sqrt3}{24}\right) = \frac{486\pi}{5}\left(\frac23 - \frac{3\sqrt3}{8}\right)$$

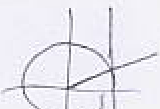
$z = \sqrt{3r^2}$

$z = \sqrt{3}\, r$

$\rho\cos\phi = \sqrt3\,\rho\sin\phi$

$\tan\phi = \frac{1}{\sqrt3}$

$\phi = \frac{\pi}{6}$

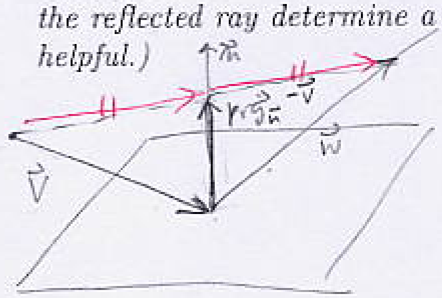


Cylindrical:

proj. to xy

$$\int_0^{2\pi}\int_0^{\frac32}\int_{\sqrt3 r}^{\sqrt{9-r^2}} r^2\cdot r\, dz\, dr\, d\theta = 2\pi\int_0^{\frac32} \underbrace{r^3\left(\sqrt{9-r^2} - \sqrt3 r\right)}_{\text{difficult}}$$

Bonus (10pts) A ray of light, represented by the line $x = 2 - t$, $y = 4 + 2t$, $z = -3 - 3t$ reflects off the mirror represented by the plane $x - y + 2z = 10$ at point $P = (4, 0, 3)$. Find parametric equations of the line that represents the reflected ray. *(Hints: the ray and the reflected ray determine a plane that is perpendicular to the mirror. Vector projection is helpful.)*



$\vec{v} = \langle -1, 2, -3\rangle$

Project $-\vec{v} = \langle 1, -2, 3\rangle$

to $\vec{n} = \langle 1, -1, 2\rangle$

$$\mathrm{proj}_{\vec n} -\vec{v} = \frac{-\vec v\cdot\vec n}{\vec n\cdot\vec n}\,\vec n = \frac{+1+2+6}{1+1+4}\langle 1,-1,2\rangle = \frac{-9}{6}\langle 1,-1,2\rangle = -\frac32\langle 1,-1,2\rangle$$

From picture we see

$\vec{w} = \mathrm{proj}_{\vec n}\,-\vec{v} + \left(\vec{v} + \mathrm{proj}_{\vec n}\,-\vec{v}\right)$

$= 2\,\mathrm{proj}_{\vec n}(-\vec v) + \vec v$

$= 2\cdot\left(-\frac32\right)\langle 1,-1,2\rangle + \langle -1, 2, -3\rangle$

$= \langle -3, -3, 6\rangle + \langle -1, 2, -3\rangle = \langle 2, -1, 3\rangle$

Param. equ'n

$x = 4 + 2t$

$y = -t$

$z = 3 + 3t$