Calculus 3 — Exam 5
MAT 309, Fall 2013 — D. Ivanšić

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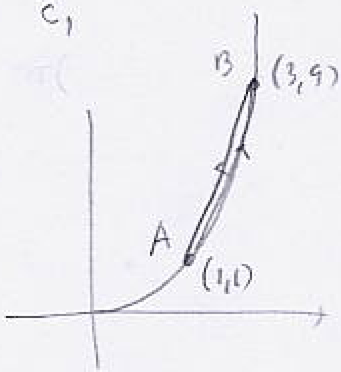
1. (10pts) Let $f(x,y) = \dfrac{x^3}{y^2}$, and let $\mathbf{F} = \nabla f$. Apply the fundamental theorem for line integrals to answer:
a) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is part of the parabola $y = x^2$ from $(1,1)$ to $(3,9)$? How about if C is a straight line segment from $(1,1)$ to $(3,9)$?
b) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the circle centered at $(3,4)$ with radius 2?

a) $\int_{C_1} \vec{F} \cdot d\vec{r} = f(B) - f(A) = \dfrac{3^3}{9^2} - \dfrac{1^3}{1^2} = \dfrac{27}{81} - 1 = -\dfrac{2}{3}$ $\quad \int_{C_2} \vec{F} \cdot d\vec{r}$ is same, 2



b) $\int_C \vec{F} \cdot d\vec{r} = 0$
(integral of a gradient field over a closed curve is 0)

2. (12pts) Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x,y,z) = \langle z^2 - 4y^2, 4x^2 - 3z^2, 3y^2 - x^2 \rangle$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - 4y^2 & 4x^2 - 3z^2 & 3y^2 - x^2 \end{vmatrix} = \langle 6y + 6z, -(-2x - 2z), 8x - (-8y) \rangle$$

$$= \langle 6(y+z), 2(x+z), 8(x+y) \rangle$$

$$\text{div}\,\vec{F} = \underbrace{\frac{\partial}{\partial x}(z^2 - 4y^2)}_{=0} + \underbrace{\frac{\partial}{\partial y}(4x^2 - 3z^2)}_{=0} + \underbrace{\frac{\partial}{\partial z}(3y^2 - x^2)}_{=0} = 0$$

3. (14pts) A surface is parametrized by $\mathbf{r}(u,v) = \langle u^2, v^2, u+v\rangle$. Find the equation of the tangent plane to this surface at the point where $(u,v) = (2,-3)$.

$\vec{r}_u = \langle 2u, 0, 1\rangle \quad \vec{r}_u(2,-3) = \langle 4,0,1\rangle \qquad \vec{r}(2,-3) = (4,9,-1)$

$\vec{r}_v = \langle 0, 2v, 1 \quad \vec{r}_v(2,-3) = \langle 0,-6,1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 1 \\ 0 & -6 & 1 \end{vmatrix} = \langle 0-(-6), -(4-0), -24\rangle = \langle 6,-4,-24\rangle$$
$$= 2\langle 3,-2,-12\rangle$$

Use $\vec{n} = \langle 3,-2,-12\rangle$

Plane: $3(x-4) - 2(y-9) - 12(z+1) = 0$

$3x - 2y + 12z = 6$

4. (20pts) One of the two vectors fields below is not a gradient field, and the other one is (curl detects it). Identify which is which, and find the potential function for the one that is.

$\mathbf{F}(x,y,z) = \langle 2x\sin z + y^3e^x, 3y^2e^x + \cos z, x^2\cos z - y\sin z\rangle \qquad \mathbf{G}(x,y,z) = \langle x^2, y^2, yz^2\rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\sin z + y^3e^x & 3y^2e^x + \cos z & x^2\cos z - y\sin z \end{vmatrix}$$

$= \langle -\sin z - (-\sin z), -(2x\cos z - 2x\cos z), 3y^2e^x - 3y^2e^x\rangle$

$= \vec{0}$ so $\vec{F}$ is conservative (being defined on $\mathbb{R}^3$)

$\frac{\partial f}{\partial x} = 2x\sin z + y^3e^x$

$\Rightarrow f = x^2\sin z + y^3e^x + g(y,z)$

$3y^2e^x + \cos z = \frac{\partial f}{\partial y} = 3y^2e^x + \frac{\partial g}{\partial y}$

$\Rightarrow \frac{\partial g}{\partial y} = \cos z$, so $g = y\cos z + h(z)$

$f = x^2\sin z + y^3e^x + y\cos z + h(z)$

$x^2\cos z - y\sin z = \frac{\partial f}{\partial z} = x^2\cos z - y\sin z + h'(z)$

$$\text{curl}\,\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & yz^2 \end{vmatrix}$$

$= \langle z^2 - 0, -(0-0), 0-0\rangle = \langle z^2,0,0\rangle \neq \vec{0}$

so $\vec{G}$ cannot be conservative

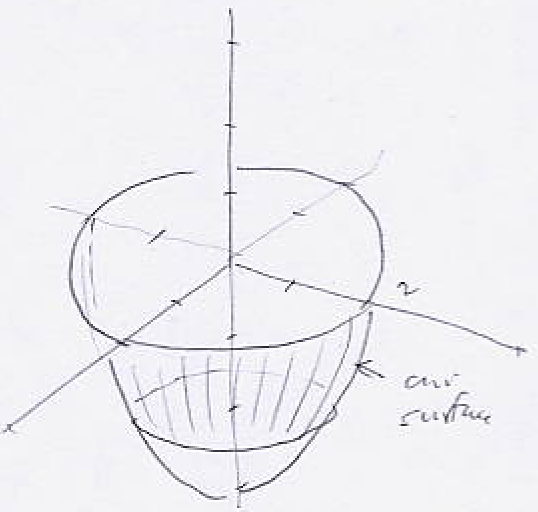
$\Rightarrow h'(z) = 0$, so $h(z) = C$

$f(x,y,z) = x^2\sin z + y^3e^x + y\cos z$

5. (26pts) Consider the part of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{9}=1$ between the planes $z=-2$ and $z=0$.

a) Draw the surface, parametrize it and specify the planar region D where your parameters come from.

b) Set up the iterated integral that gives the area of the surface. Simplify the set-up, but do not evaluate the integral.



Parametrization

$x=u$

$y=v$

$z=-3\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}$

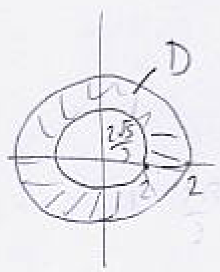
$\frac{z^2}{9}=1-\frac{x^2}{4}-\frac{y^2}{4}$

$z^2=9\left(1-\frac{x^2}{4}-\frac{y^2}{4}\right)$

$z=\pm 3\sqrt{1-\frac{x^2}{4}-\frac{y^2}{4}}$

− gives the bottom half.

Project to xy plane:



When $z=0$

$\frac{x^2}{4}+\frac{y^2}{4}=1$ circle of radius 2

$x^2+y^2=4$

When $z=-2$

$\frac{x^2}{4}+\frac{y^2}{4}+\frac{4}{9}=1$

$\frac{x^2}{4}+\frac{y^2}{4}=\frac{5}{9}$ circle radius

$x^2+y^2=\frac{20}{9}$ $\sqrt{\frac{20}{9}}=\frac{2\sqrt{5}}{3}$

b)

$\vec{r}_u=\left\langle 1,0,-\frac{3}{2\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}}\cdot-\frac{2u}{4}\right\rangle$

$=\left\langle 1,0,\frac{3u}{4\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}}\right\rangle$

$\vec{r}_v=\left\langle 0,1,-\frac{3}{2\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}}\cdot-\frac{2v}{4}\right\rangle$

$=\left\langle 0,1,\frac{3v}{4\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}}\right\rangle$

$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & \square\\ 0 & 1 & \square\end{vmatrix}=\left\langle-\frac{3u}{4\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}},-\frac{3v}{4\sqrt{1-\frac{u^2}{4}-\frac{v^2}{4}}},1\right\rangle$

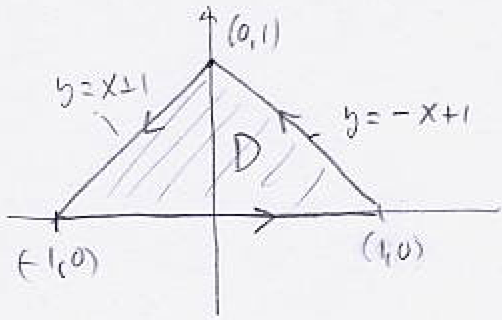
$|\vec{r}_u\times\vec{r}_v|=\sqrt{\frac{9u^2}{16\left(1-\frac{u^2}{4}-\frac{v^2}{4}\right)}+\frac{9v^2}{16\left(1-\frac{u^2}{4}-\frac{v^2}{4}\right)}+1}$

$=\sqrt{\frac{16+5u^2+5v^2}{16-4u^2-4v^2}}$

Use polar coord

$\iint_D\sqrt{\frac{16+5(u^2+v^2)}{16-4(u^2+v^2)}}\,dA=\int_0^{2\pi}\int_{\frac{2\sqrt{5}}{3}}^{2}\sqrt{\frac{16+5r^2}{16-4r^2}}\,r\,dr\,d\theta$

6. (18pts) Use Green's theorem to find the line integral $\int_C x^3\,dx + xy\,dy$, where C is the triangle from $(-1,0)$ to $(1,0)$ to $(0,1)$ to $(-1,0)$.



$$\int_C x^3dx + xy\,dy = \iint_D \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}x^3\,dA$$

Green's thm

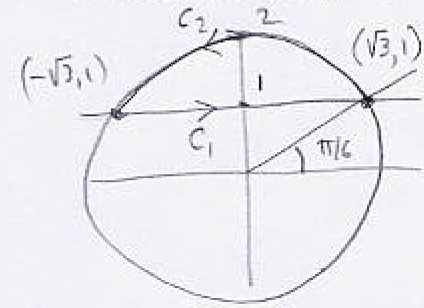
$$= \iint_D y\,dA = \int_0^1\int_{y-1}^{-y+1} y\,dx\,dy = \int_0^1 y(-y+1-(y-1))\,dy$$

type 2 region

$$= \int_0^1 y(2-2y)\,dy = \int_0^1 2y-2y^2\,dy = \left(y^2-\frac{2}{3}y^3\right)\Big|_0^1$$

$$= 1-\frac{2}{3} = \frac{1}{3}$$

Bonus. (10pts) Use Green's theorem to find the area enclosed by the circle $x^2+y^2=4$ that is above the line $y=1$.



$$\text{Area} = \iint 1\,dA = \iint \frac{\partial}{\partial x}x\,dA = \int_{C_1\cup C_2} 0\,dx + x\,dy$$

Green's thm

$x^2+1^2=4$

$x=\pm\sqrt{3}$

C_1 $x=t$ $x'=1$ $\int_0^1 \langle 0,t\rangle\cdot\langle 1,0\rangle\,dt = 0$

$y=1$ $y'=0$

$-\sqrt{3}\le t\le\sqrt{3}$

C_2 $x=\cos t$ $x'=-\sin t$ $\int_{\pi/6}^{5\pi/6}\langle 0,\cos t\rangle\cdot\langle -\sin t,\cos t\rangle\,dt$

$y=\sin t$ $y'=\cos t$

$\frac{\pi}{6}\le t\le\frac{5\pi}{6}$

$$= \int_{\pi/6}^{5\pi/6}\cos^2t\,dt = \int_{\pi/6}^{5\pi/6}\frac{1}{2}(1+\cos 2t)\,dt = \frac{1}{2}\left(\frac{5\pi}{6}-\frac{\pi}{6}+\frac{\sin 2t}{2}\Big|_{\pi/6}^{5\pi/6}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{3}+\frac{1}{2}\left(\underbrace{\sin\frac{5\pi}{3}}_{-\frac{\sqrt{3}}{2}}-\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}\right)\right) = \frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}-\frac{\sqrt{3}}{4}$$