Calculus 3 — Exam 4
MAT 309, Fall 2013 — D. Ivanšić

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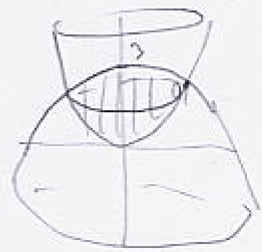
1. (18pts) Use cylindrical coordinates to find the volume of the region E enclosed by the paraboloids $z = x^2 + y^2$ and $z = 3 - \frac{1}{2}(x^2 + y^2)$. Sketch the region E.

$x^2+y^2=r^2$

$z=r^2$

$z=3-\frac{1}{2}r^2$



Intersection:

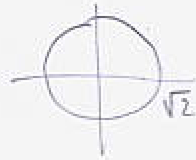
$r^2=3-\frac{1}{2}r^2$

$\frac{3}{2}r^2=3$

$r^2=2$

$r=\sqrt{2}$

Projection to xy plane:



$V=\iiint_E 1\,dV$

$= \underbrace{\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{3-\frac{1}{2}r^2} r\,dz\,dr\,d\theta}_{\text{no } \theta \text{ here}}$

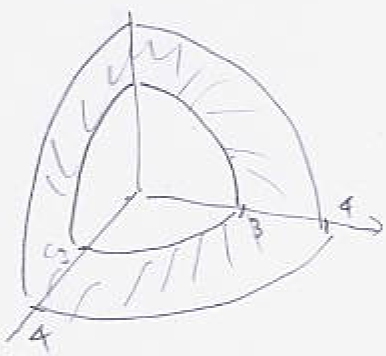
$=2\pi\int_0^{\sqrt{2}} r\left(3-\frac{1}{2}r^2-r^2\right)dr$

$=2\pi\int_0^{\sqrt{2}} 3r-\frac{3}{2}r^3\,dr = 2\pi\left(\frac{3r^2}{2}-\frac{3}{8}r^4\right)\Big|_0^{\sqrt{2}}$

$=2\pi\left(3-\frac{3}{2}\right)=2\pi\cdot\frac{3}{2}=3\pi$

2. (18pts) Use spherical coordinates to find $\iiint_E xz\,dV$, where E is the part of the first octant that is inside the sphere $x^2 + y^2 + z^2 = 16$, and outside the sphere $x^2 + y^2 + z^2 = 9$. Sketch the region E.

radius = 4 radius = 3



$\iiint_E xz\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_3^4 \rho\sin\phi\cos\theta\,\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$=\int_0^{\pi/2}\int_0^{\pi/2}\int_3^4 \rho^4\sin^2\phi\cos\phi\cos\theta\,d\rho\,d\phi\,d\theta$

$=\int_0^{\pi/2}\cos\theta\,d\theta\cdot\int_0^{\pi/2}\sin^2\phi\cos\phi\,d\phi\cdot\int_3^4\rho^4\,d\rho$

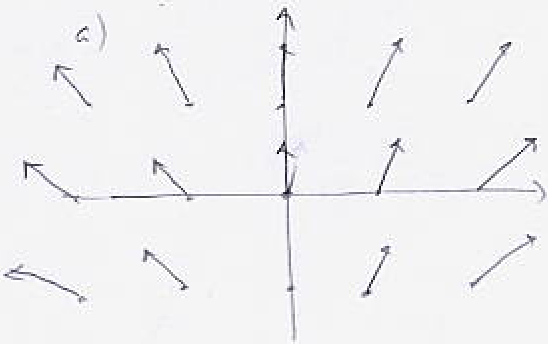
$=\sin\theta\Big|_0^{\pi/2}\cdot\frac{1}{3}\sin^3\phi\Big|_0^{\pi/2}\cdot\frac{\rho^5}{5}\Big|_3^4$

$=(1-0)\frac{1}{3}(1-0)\cdot\frac{1}{5}(4^5-3^5)=\frac{4^5-3^5}{15}=\frac{781}{15}$

$4^5=2^{10}=1024 \quad 3^5=243$

3. (14pts) Let $\mathbf{F}(x, y) = \langle x, 3\rangle$.
a) Roughly draw the vector field $\mathbf{F}(x, y)$, scaling the vectors for a better picture.
b) Guess a function $f(x, y)$ so that $\mathbf{F} = \nabla f$.
c) How could you have roughly done a) without evaluating the vector field at various points?

a)



b) $\nabla f = \langle f_x, f_y\rangle = \langle x, 3\rangle$

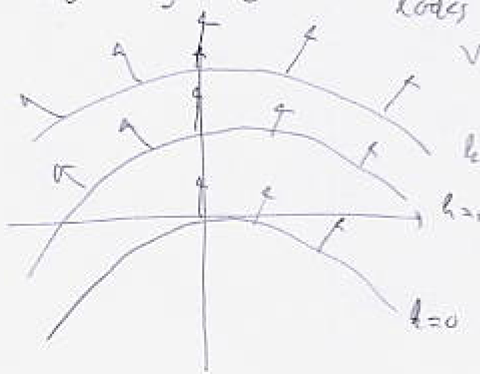
$f = \frac{1}{2}x^2 + 3y$

c) ∇f is perpendicular to level curves of f. The level curves are:

$\frac{1}{2}x^2 + 3y = k$

$y = \frac{k}{3} - \frac{1}{6}x^2$

looks like our vector field



4. (18pts) In both cases set up and simplify the set-up, but do not evaluate the integral.
a) $\int_C x^2 - y^2 + z^2\, ds$, where C is the line segment from $(0, 0, 1)$ to $(1, -3, 3)$.
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x, y) = \langle xe^y, ye^x\rangle$, where C is the circle of radius 5 centered at the origin.

a) $\vec{v} = \langle 1-0, -3-0, 3-1\rangle$
$= \langle 1, -3, 2\rangle$

$x = t \quad x' = 1 \quad t \text{ in } [0,1]$
$y = -3t \quad y' = -3$
$z = 1 + 2t \quad z' = 2$

$\int_C x^2 - y^2 + z^2\, ds = \int_0^1 \left(t^2 - (-3t)^2 + (1+2t)^2\right)\sqrt{1^2 + (-3)^2 + 2^2}\, dt$

$= \int_0^1 (t^2 - 9t^2 + 1 + 4t + 4t^2)\sqrt{14}\, dt$

$= \int_0^1 \sqrt{14}(-4t^2 + 4t + 1)\, dt$

b) $x = 5\cos t \quad x' = -5\sin t \quad t \text{ in } [0, 2\pi]$
$y = 5\sin t \quad y' = 5\cos t$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 5\cos t\, e^{5\sin t}, 5\sin t\, e^{5\cos t}\rangle \cdot \langle -5\sin t, 5\cos t\rangle\, dt$

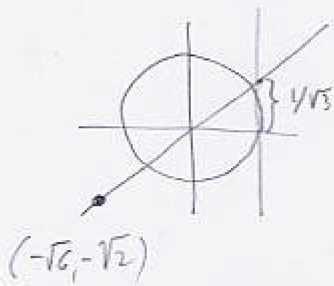
$= \int_0^{2\pi} -25\sin t\cos t\, e^{5\sin t} + 25\sin t\cos t\, e^{5\cos t}\, dt$

$= \int_0^{2\pi} 25\sin t\cos t\left(e^{5\cos t} - e^{5\sin t}\right) dt$

5. (12pts) Find the cylindrical and spherical coordinates of the point whose cartesian coordinates are $(-\sqrt{6}, -\sqrt{2}, 2\sqrt{2})$.

$$r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$$
$$= \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{-\sqrt{2}}{-\sqrt{6}} = \frac{1}{\sqrt{3}}$$



$\theta = \frac{\pi}{6} + \pi$ since x, y are in 3rd quad.

$\theta = \frac{7\pi}{6}$

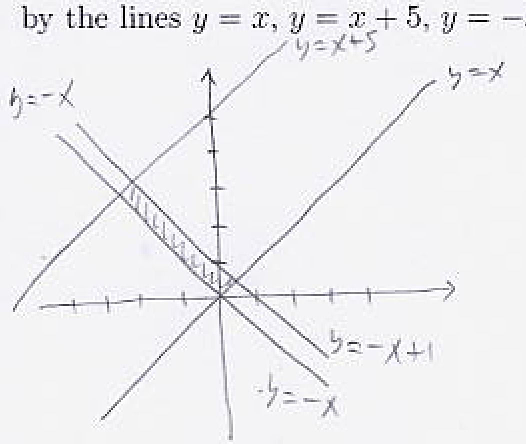
Cylindrical: $\left(2\sqrt{2}, \frac{7\pi}{6}, 2\sqrt{2}\right)$

$$\rho = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2 + (2\sqrt{2})^2}$$
$$= \sqrt{6+2+8} = \sqrt{16} = 4$$

$$z = \rho\cos\phi$$
$$2\sqrt{2} = 4\cos\phi$$
$$\cos\phi = \frac{\sqrt{2}}{2}, \text{ so } \phi = \frac{\pi}{4}$$

Spherical: $\left(4, \frac{7\pi}{6}, \frac{\pi}{4}\right)$

6. (20pts) Use change of variables to find $\iint_D y\,dA$, if D is the rectangle that is bounded by the lines $y = x$, $y = x + 5$, $y = -x$, $y = -x + 1$. Sketch the rectangle.



Set: $\begin{cases} y = x + u \\ y = -x + v \end{cases}$ $\xrightarrow{\text{subtract}}$ $0 = 2x + u - v$

$\boxed{x = \frac{1}{2}(-u+v)}$

$2y = u + v$

$\boxed{y = \frac{1}{2}(u+v)}$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

region S: $0 \le u \le 5$, $0 \le v \le 1$

$$\iint_D y\,dA = \iint_S \frac{1}{2}(u+v)\cdot \overbrace{\left|-\frac{1}{2}\right|}^{=\frac{1}{2}}\,dA$$

$$= \int_0^5 \int_0^1 \frac{1}{4}(u+v)\,dv\,du = \frac{1}{4}\int_0^5 u(1-0) + \frac{v^2}{2}\Big|_0^1\,du$$

$$= \frac{1}{4}\int_0^5 u + \frac{1}{2}\,du = \frac{1}{4}\left(\frac{u^2}{2}\Big|_0^5 + \frac{1}{2}(5-0)\right) = \frac{1}{4}\left(\frac{25}{2} + \frac{5}{2}\right) = \frac{15}{4}$$

Bonus. (10pts) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\Phi)}$, where x, y, z are functions that convert spherical coordinates to cartesian. What do you expect to get?

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{array}{c|ccc|} & \partial/\partial\rho & \partial/\partial\theta & \partial/\partial\phi \\ x & \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ y & \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ z & \cos\phi & 0 & -\rho\sin\phi \end{array}$$

expand by 3rd row

$$\cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$

$$-\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= -\rho^2\sin\phi\cos^2\phi\,(\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) - \rho^2\sin^3\phi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$$

$$= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) = -\rho^2\sin\phi$$

Since $0 \le \phi \le \pi$, $\sin\phi \ge 0$ so $|-\rho^2\sin\phi| = \rho^2\sin\phi$, exactly the distortion factor that comes up in spherical coordinates