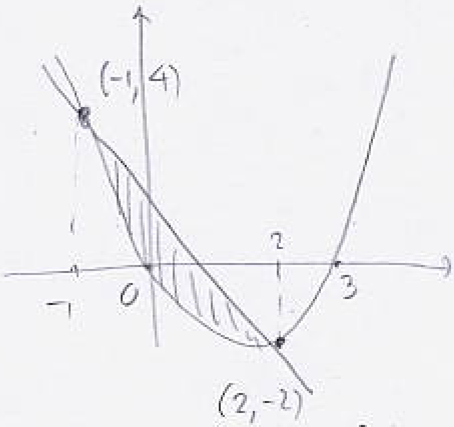
Calculus 3 — Exam 3 MAT 309, Fall 2013 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (16pts) Find $\iint_D x\,dA$ if D is the region bounded by $y = 2-2x$ and $y = x^2-3x$. Sketch the region of integration first.



$$\begin{cases} y = 2-2x \\ y = x^2-3x \end{cases}$$

$$x^2-3x = 2-2x$$

$$x^2-x-2=0$$

$$(x-2)(x+1)=0$$

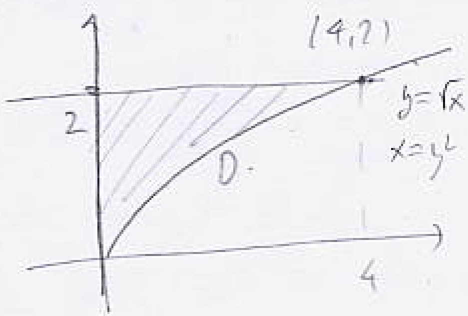
$$x = 2, -1$$

$$\iint_D x\,dA = \int_{-1}^{2}\int_{x^2-3x}^{2-2x} x\,dy\,dx = \int_{-1}^{2} x\left(2-2x-(x^2-3x)\right)dx = \int_{-1}^{2} x(-x^2+x+2)\,dx$$

$$= \int_{-1}^{2} -x^3+x^2+2x\,dx = \left(-\frac{x^4}{4}+\frac{x^3}{3}+x^2\right)\Big|_{-1}^{2} = -\frac{1}{4}(16-1)+\frac{1}{3}(8+1)+4-1$$

$$= -\frac{15}{4}+3+3 = \frac{9}{4}$$

2. (18pts) Let D be the region bounded by the curves $x=0$, $y=2$ and $y=\sqrt{x}$. Sketch the region and set up $\iint_D (y^3+1)^5\,dA$ as iterated integrals in both orders of integration. Then evaluate the double integral using the easier order.



$$\iint_D (y^3+1)^5\,dy = \int_0^4\int_{\sqrt{x}}^{2} (y^3+1)^5\,dy\,dx \leftarrow \text{hard to integrate by } y$$

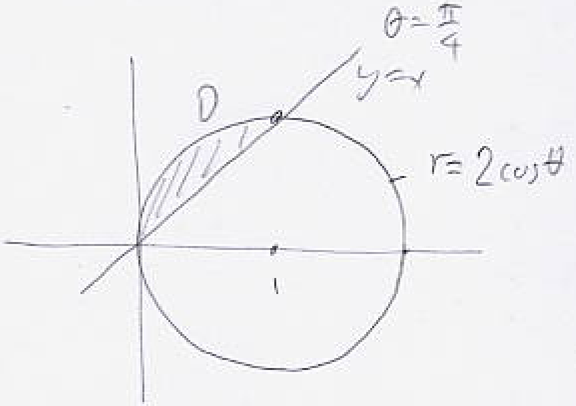
OR

$$= \int_0^2\int_0^{y^2} (y^3+1)^5\,dx\,dy \leftarrow \text{easier}$$

$$= \int_0^2 (y^3+1)^5(y^2-0)\,dy = \int_0^2 y^2(y^3+1)^5\,dy = \left[\begin{array}{l} u = y^3+1 \\ du = 3y^2\,dy \\ \frac{1}{3}du = y^2\,dy \end{array}\right. \quad \begin{array}{l} y=2,\ u=9 \\ y=0,\ u=1 \end{array}$$

$$= \int_1^9 u^5\frac{1}{3}\,du = \frac{1}{3}\cdot\frac{u^6}{6}\Big|_1^9 = \frac{1}{18}(9^6-1)$$

3. (18pts) Use polar coordinates to find the area of the region that is inside the circle $(x-1)^2 + y^2 = 1$ and above the line $y = x$. Sketch the region of integration first.



$$\text{Area} = \iint 1\, dA = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} 1 \cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{r^2}{2} \Big|_0^{2\cos\theta} d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} 2 \cdot \frac{1+\cos 2\theta}{2}\, d\theta = \left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2}(\underset{0}{\underbrace{\sin\pi}} - \sin\frac{\pi}{2}) = \frac{\pi}{4} - \frac{1}{2}$$

4. (20pts) Find and classify the local extremes for $f(x, y) = x^3 + 3x^2y - y^3 + 9y$.

$$\nabla f = \langle 3x^2 + 6xy,\ 3x^2 - 3y^2 + 9 \rangle$$

$$D = \begin{vmatrix} 6x+6y & 6x \\ 6x & -6y \end{vmatrix}$$

Critical pts: 1st eq

$$\begin{cases} 3x^2 + 6xy = 0 & \mid \div 3 \\ 3x^2 - 3y^2 + 9 = 0 & \mid \div 3 \end{cases}$$

$$\begin{cases} x(x+2y) = 0 \\ x^2 - y^2 + 3 = 0 \end{cases}$$

1st eq gives:

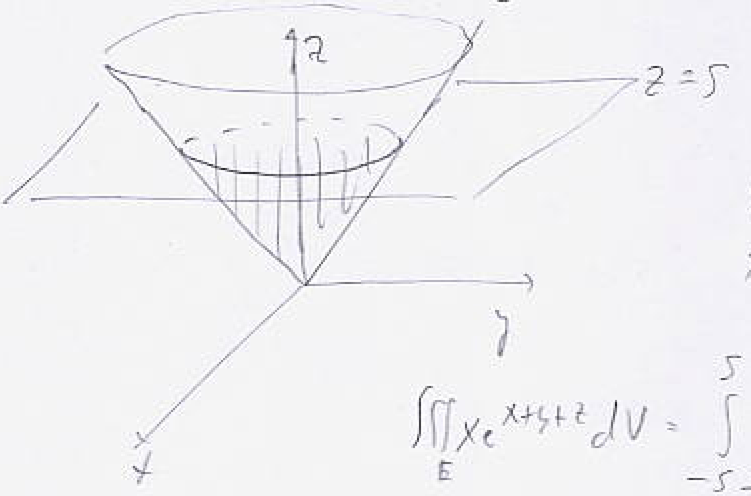
$x = 0$ OR $x = -2y$

put in 2nd $\Rightarrow -y^2 + 3 = 0$, $y = \pm\sqrt{3}$

put in 2nd $\Rightarrow (-2y)^2 - y^2 + 3 = 0$, $-3y^2 + 3 = 0$, $y^2 = -1$ no real sol.

Candidates	D(x,y)	
$(0, \sqrt{3})$	$\begin{vmatrix} 6\sqrt{3} & 0 \\ 0 & -6\sqrt{3} \end{vmatrix} < 0$	so get a saddle point
$(0, -\sqrt{3})$	$\begin{vmatrix} 6(-\sqrt{3}) & 0 \\ 0 & 6\sqrt{3} \end{vmatrix} < 0$	so get a saddle point

5. (12pts) Set up the triple iterated integral for $\iiint_E xe^{x+y+z}\,dV$, where E is the region bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 5$. Sketch the region of integration first. Do not evaluate the integral.



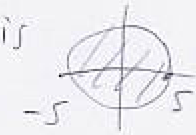
$z = \sqrt{x^2+y^2}$

$z = 5$

$\sqrt{x^2+y^2} = 5$

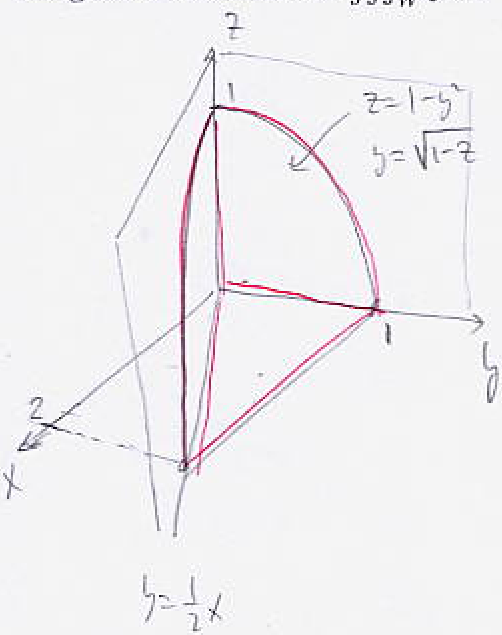
$x^2+y^2 = 25$

Projection of region is

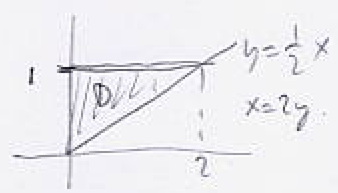


$$\iiint_E xe^{x+y+z}\,dV = \int_{-5}^{5}\int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}}\int_{\sqrt{x^2+y^2}}^{5} xe^{x+y+z}\,dz\,dy\,dx$$

6. (16pts) Sketch the region E that is in the first octant ($x, y, z \geq 0$), and bounded by the plane $y = \frac{1}{2}x$ and the parabolic cylinder $z = 1 - y^2$. Then write the two iterated triple integrals that stand for $\iiint_W f\,dV$ which end in $dz\,dx\,dy$ and $dy\,dz\,dx$.

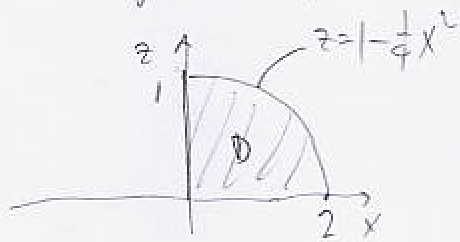


Proj. to xy plane



$$\iint_D \int_0^{1-y^2} f\,dz\,dA = \int_0^1\int_0^{2y}\int_0^{1-y^2} f\,dz\,dx\,dy$$

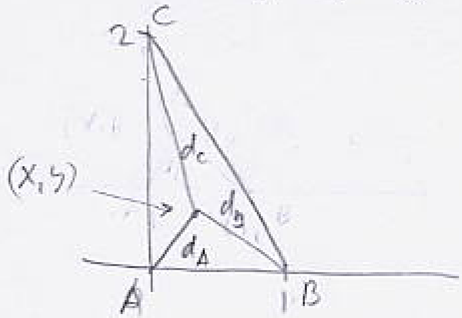
Proj. to xz-plane



$$\iint_D \int_{\frac{1}{2}x}^{\sqrt{1-z}} f\,dy\,dA = \int_0^2\int_0^{1-\frac{1}{4}x^2}\int_{\frac{1}{2}x}^{\sqrt{1-z}} f\,dy\,dz\,dx$$

$$\begin{cases} z = 1-y^2 \\ y = \frac{1}{2}x \end{cases} \Rightarrow z = 1 - \frac{1}{4}x^2$$

Bonus (10pts) Let $A = (0,0)$, $B = (1,0)$ and $C = (0,2)$ and let d_A, d_B and d_C represent the distance from a point (x, y) to A, B and C, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2 + d_C^2$ among all points (x, y) in the triangle ABC (edges are included).



$$f(x,y) = d_A^2 + d_B^2 + d_C^2$$
$$= x^2+y^2+(x-1)^2+y^2+x^2+(y-2)^2$$
$$= 3x^2+3y^2-2x-4y+5$$

$\nabla f = \langle 6x-2, 6y-4 \rangle$

Critical pt.

$\begin{cases} 6x-2=0 & x=\frac{1}{3} \\ 6y-4=0 & y=\frac{2}{3} \end{cases}$

Boundary:

$\boxed{AB}$ $x=t$, $y=0$, t in $[0,1]$ $f(t,0) = 3t^2-2t+5$

$f'(t) = 6t-2$, $t=\frac{1}{3}$

$6t-2=0$, $t=\frac{1}{3}$

$(\frac{1}{3}, 0)$

$\boxed{BC}$ $y = 2-2x$

$x = t$, $y = 2-2t$, t in $[0,1]$

$f(t, 2-2t) = 3t^2+3(2-2t)^2-2t-4(2-2t)+5$

$g'(t) = 6t+6(2-2t)(-2)-2+8$

$= 30t-18$

$30t-18=0$, $t=\frac{18}{30}=\frac{3}{5}$

$(\frac{3}{5}, \frac{4}{5})$

$\boxed{AC}$ $x=0$, $y=t$, t in $[0,2]$ $f(0,t) = 3t^2-4t+5$

$g'(t) = 6t-4$, $t=\frac{2}{3}$

$6t-4=0$

$(0, \frac{2}{3})$

(x,y)	$f(x,y)$	
$(\frac{1}{3}, \frac{2}{3})$	$3\cdot\frac{1}{9}+3\cdot\frac{4}{9}-\frac{2}{3}-\frac{8}{3}+5=\frac{10}{3}$	min
$(\frac{1}{3}, 0)$	$3\cdot\frac{1}{9}-\frac{2}{3}+5=\frac{14}{3}$	
$(\frac{3}{5}, \frac{4}{5})$	$3\cdot\frac{9}{25}+3\cdot\frac{16}{25}-\frac{6}{5}-\frac{16}{5}+5=\frac{90}{25}=\frac{18}{5}$	
$(0, \frac{2}{3})$	$3\cdot\frac{4}{9}-\frac{8}{3}+5=\frac{11}{3}$	
$(0,0)$	5	
$(1,0)$	$3-2+5=6$	
$(0,2)$	$12-8+5=9$	max

$27+48-30-80+125=90$

$\frac{10}{3} \overset{?}{<} \frac{90}{25}$

$250 < 270$ yes