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| **Calculus 3 — Exam 2**<br>**MAT 309, Fall 2013 — D. Ivanšić** |

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*Show all your work!*

**1.** (22pts) Let $T(x, y) = \dfrac{y}{x^2}$.

a) Find the domain of $T$.

b) Sketch the contour map for the function, drawing level curves for levels $k = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Note the domain on the picture.

c) Suppose $T$ represents temperature in degrees Celsius in the plane, and a freezing bug located at $(2, -4)$ wishes to move to a point with a higher temperature. In what direction should it start moving to achieve the greatest increase in temperature? What is the directional derivative in that direction?

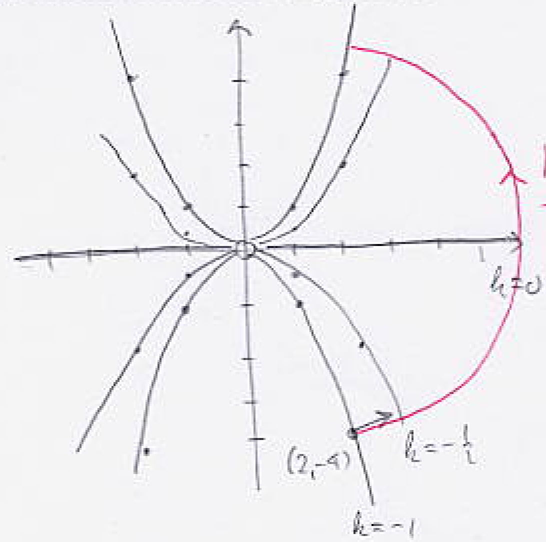
d) Draw a path the bug would take in order to reach a point with temperature 1°C if it always moves in the direction of the greatest increase of temperature.

a) Can't have $x=0$

Domain $\{(x,y) \mid x \neq 0\}$

b) $\dfrac{y}{x^2} = k$

$y = kx^2$ parabolas




path of bug - perp. to level curves

domain = everything except y-axis

c) Bug should start moving in direction of $\nabla f(2,-4)$

$\nabla f = \left\langle -\dfrac{2y}{x^3}, \dfrac{1}{x^2} \right\rangle$

$\nabla f(2,-4) = \left\langle -\dfrac{2\cdot(-4)}{8}, \dfrac{1}{4} \right\rangle = \left\langle 1, \dfrac{1}{4} \right\rangle$

Directional derivative is $|\nabla f|$

$= \sqrt{1+\left(\frac{1}{4}\right)^2} = \sqrt{1+\frac{1}{16}} = \sqrt{\frac{17}{16}} = \dfrac{\sqrt{17}}{4}$

**2.** (10pts) Find the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{25} + \dfrac{z^2}{16} = 1$ at the point $(\sqrt{2}, -\frac{5}{2}, -2)$. Simplify the equation to standard form.

$F(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{25} + \dfrac{z^2}{16} - 1$

$\nabla F = \left\langle \dfrac{2x}{4}, \dfrac{2y}{25}, \dfrac{2z}{16} \right\rangle = \left\langle \dfrac{x}{2}, \dfrac{2y}{25}, \dfrac{z}{8} \right\rangle$

$\nabla F(\sqrt{2}, -\frac{5}{2}, -2) = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{2\cdot(-\frac{5}{2})}{25}, \dfrac{-2}{8} \right\rangle$

$= \left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{1}{5}, -\dfrac{1}{4} \right\rangle$

Eq. of plane:

$\dfrac{\sqrt{2}}{2}(x-\sqrt{2}) + \left(-\dfrac{1}{5}\right)\left(x+\dfrac{5}{2}\right) - \dfrac{1}{4}(z+2) = 0$

$\dfrac{\sqrt{2}}{2}x - 1 - \dfrac{1}{5}x - \dfrac{1}{2} - \dfrac{1}{4}z - \dfrac{1}{2} = 0$

$\dfrac{\sqrt{2}}{2}x - \dfrac{1}{5}x - \dfrac{1}{4}z = 2$

**3.** (18pts) Let $B = \dfrac{x^2+y^2}{x+1}$, $x = \cos u + \sin v$, $y = \sin u \cos v$. Use the chain rule to find $\dfrac{\partial B}{\partial v}$ when $u = \frac{\pi}{4}$, $v = \pi$.

$$\frac{\partial B}{\partial v} = \frac{\partial B}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial B}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{2x(x+1)-(x^2+y^2)\cdot 1}{(x+1)^2}\cdot \cos v + \frac{2y}{x+1}\cdot \sin u(-\sin v)$$

$$= \frac{x^2-y^2+2x}{(x+1)^2}\cos v - \frac{2y}{x+1}\sin u \sin v$$

when $u=\frac{\pi}{4}$, $v=\pi$

$x=\frac{\sqrt2}{2}$ $y=-\frac{\sqrt2}{2}$

$$\frac{\partial B}{\partial v} = \frac{\frac12-\frac12+2\frac{\sqrt2}{2}}{\left(\frac{\sqrt2}{2}+1\right)^2}\cdot(-1) - \frac{2\left(-\frac{\sqrt2}{2}\right)}{\frac{\sqrt2}{2}+1}\cdot\frac{\sqrt2}{2}\cdot 0$$

$$= -\frac{\sqrt2}{\left(\frac{\sqrt2}{2}+1\right)^2}$$

**4.** (16pts) The body surface area $S$ in $\mathrm{m}^2$ can be calculated from a person's weight $w$ in kg and height $h$ in cm using the formula $S = \dfrac{\sqrt{wh}}{60}$. Use differentials to estimate the change in body surface area of a woman who weighs 64kg and is 169cm tall if her height increases by 2cm and weight decreases by 0.5kg. Substitute all the numbers, and simplify what you can, but stop when the numbers get hairy. (Note: $13^2 = 169$.)

$$dS = \frac{\partial S}{\partial w}dw + \frac{\partial S}{\partial h}dh \qquad S = \frac{\sqrt{w}\cdot\sqrt{h}}{60}$$

$$\frac{\partial S}{\partial w} = \frac{\sqrt h}{2\sqrt w\cdot 60} \qquad \frac{\partial S}{\partial h} = \frac{\sqrt w}{2\sqrt h\cdot 60}$$

Eval. at $w=64$, $h=169$, $dw=-0.5$, $dh=2$cm

$$dS = \frac{\sqrt{169}}{120\sqrt{64}}\cdot(-0.5) + \frac{\sqrt{64}}{120\sqrt{169}}\cdot 2 = -\frac{13}{120\cdot 8}\cdot\frac12 + \frac{8\cdot 2}{\underset{15}{120}\cdot 13} = -\frac{13}{\underset{128\cdot 15}{1920}} + \frac{2}{\underset{13\cdot 15}{195}}$$

$120\cdot 13$ $\quad \frac{15\cdot 13}{\frac{45}{195}}$

$$= \frac{-13\cdot 13 + 2\cdot 120}{128\cdot 13\cdot 15} = \frac{71}{128\cdot 13\cdot 15}$$

**5.** (20pts) At a state fair, junked cars get catapulted at inital speed 25m/s and angle $\alpha$ for which $\tan\alpha = \frac{1}{2}$. Assume $g = 10$.

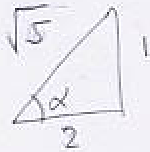
a) Find the position of the car at time $t$.

b) When does the car fall to the ground?

b) Find the horizontal distance that the car will travel.

a) $\vec{a}(t) = \langle 0, -10 \rangle$

$\vec{v}(t) = \langle 0, -10t \rangle + \vec{C}$




$\sin\alpha = \frac{1}{\sqrt{5}}$

$\cos\alpha = \frac{2}{\sqrt{5}}$

$\langle 25 \cdot \frac{2}{\sqrt{5}}, 25 \frac{1}{\sqrt{5}} \rangle = \vec{v}(0) = \vec{0} + \vec{C}$

$\vec{v}(t) = \langle 10\sqrt{5}, 5\sqrt{5} - 10t \rangle$


$\vec{r}(t) = \langle 10\sqrt{5}\, t, 5\sqrt{5}\, t - 5t^2 \rangle + \vec{D}$

$\vec{0} = \vec{r}(0) = 0 + \vec{D}$, so $\vec{D} = 0$

$\vec{r}(t) = \langle 10\sqrt{5}\, t, 5\sqrt{5}\, t - 5t^2 \rangle$

b) Hits ground when $y(t) = 0$

$5\sqrt{5}\, t - 5t^2 = 0$

$5t(\sqrt{5} - t) = 0$

$t = 0$ or $t = \sqrt{5}$

c) Need $x(\sqrt{5})$

$= 10\sqrt{5} \cdot \sqrt{5} = 50$ meters

**6.** (14pts) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ at the point $(1, 2, -1)$, if $yz^3 + xz^2 - x^2y^3 = -9$.

$$F(x,y,z) = yz^3 + xz^2 - x^2y^3 + 9$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{z^2 - 2xy^3}{3yz^2 + 2xz}$$

$$\frac{\partial z}{\partial x}(1,2) = -\frac{(-1)^2 - 2\cdot 1\cdot 8}{3\cdot 2\cdot 1 + 2\cdot 1\cdot(-1)} = -\frac{1-16}{6-2} = -\frac{-15}{4} = \frac{15}{4}$$

**Bonus** (10pts) Show that the bug in problem 1 moves along the ellipse $\frac{x^2}{36} + \frac{y^2}{18} = 1$. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla T(\mathbf{r}(t))$. *Hint: a parametrization to ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $x = a\cos t$, $y = b\sin t$.

The ellipse is parametrized by

$x = 6\cos t$ $\qquad x' = -6\sin t$

$y = \sqrt{18}\sin t = 3\sqrt{2}\sin t$ $\qquad y' = 3\sqrt{2}\cos t$

$$\nabla T = \left\langle -\frac{2y}{x^3}, \frac{1}{x^2} \right\rangle$$

$$\nabla T(6\cos t, 3\sqrt{2}\sin t) = \left\langle -\frac{2\cdot 3\sqrt{2}\sin t}{(6\cos t)^3}, \frac{1}{(6\cos t)^2} \right\rangle = \left\langle -\frac{6\sqrt{2}\sin t}{6^3\cos^3 t}, \frac{1}{6^2\cos^2 t} \right\rangle$$

$3\cdot 2$

$$= \left\langle -\frac{\sqrt{2}\cdot 6\sin t}{6^3\cos^3 t}, \frac{6\cos t}{6^3\cos^3 t} \right\rangle = \frac{\sqrt{2}}{6^3\cos^3 t}\left\langle -6\sin t, \frac{6\cos t}{\sqrt{2}} \right\rangle$$

$$= \frac{\sqrt{2}}{6^3\cos^3 t}\left\langle -6\sin t, 3\sqrt{2}\cos t \right\rangle$$

We get $\nabla T(\vec{r}(t)) = \frac{\sqrt{2}}{6^3\cos^3 t}\vec{r}'(t)$

hence $\vec{r}'(t)$ is parallel to $\nabla T(\vec{r}(t))$