Calculus 3 — Exam 1
MAT 309, Fall 2013 — D. Ivanšić

Name: Saul Ocean

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1. (16pts) Let $\mathbf{u} = \langle 1, 1, -4 \rangle$ and $\mathbf{v} = \langle 3, -1, 0 \rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - 4\mathbf{v}$, and $|\mathbf{u}|$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

a) $3\vec{u} = \langle 3, 3, -12 \rangle$

$2\vec{u} - 4\vec{v} = \langle 2, 2, -8 \rangle - \langle 12, -4, 0 \rangle$

$= \langle -10, 6, -8 \rangle$

$|\vec{u}| = \sqrt{1^2 + 1^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$

b) $\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{3^2 + (-1)^2 + 0^2}} \langle 3, -1, 0 \rangle$

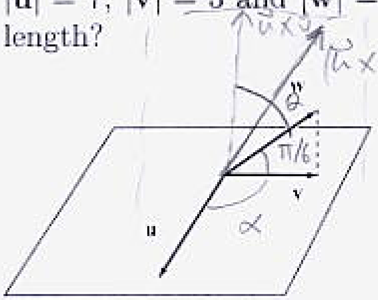
$= \frac{1}{\sqrt{10}} \langle 3, -1, 0 \rangle$

c) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{3 - 1 + 0}{\sqrt{18}\sqrt{10}}$

$= \frac{2}{\sqrt{180}} = \frac{2}{\sqrt{36 \cdot 5}} = \frac{2}{6\sqrt{5}} = \frac{1}{3\sqrt{5}}$

$\theta = \arccos \frac{1}{3\sqrt{5}}$

2. (12pts) In the picture, vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and the projection of vector $\mathbf{w}$ to the vector $\mathbf{v}$ is the vector $\mathbf{v}$. The angle between vectors $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. Suppose that $|\mathbf{u}| = 7$, $|\mathbf{v}| = 3$ and $|\mathbf{w}| = 2\sqrt{3}$. Draw the vector $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ in the picture. What is its length?



$\vec{u} \times \vec{v}$ is perp. to plane of $\vec{u}, \vec{v}$

$(\vec{u} \times \vec{v}) \times \vec{w}$ is perp. to $\vec{u} \times \vec{v}$, so, in plane,

and perp to plane of $\vec{u} \times \vec{v}$ and $\vec{w}$,

so perp. to $\vec{v}$. Thus, it is in direction of $\vec{u}$,

pointing opposite of $\vec{u}$ due to right-hand rule.

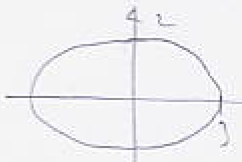
$|(\vec{u} \times \vec{v}) \times \vec{w}| = |\vec{u} \times \vec{v}||\vec{w}| \sin\theta = |\vec{u}||\vec{v}| \sin\alpha \, |\vec{w}| \sin\theta$

$= 7 \cdot 3 \sin\frac{\pi}{2} \, 2\sqrt{3} \sin\frac{\pi}{3}$

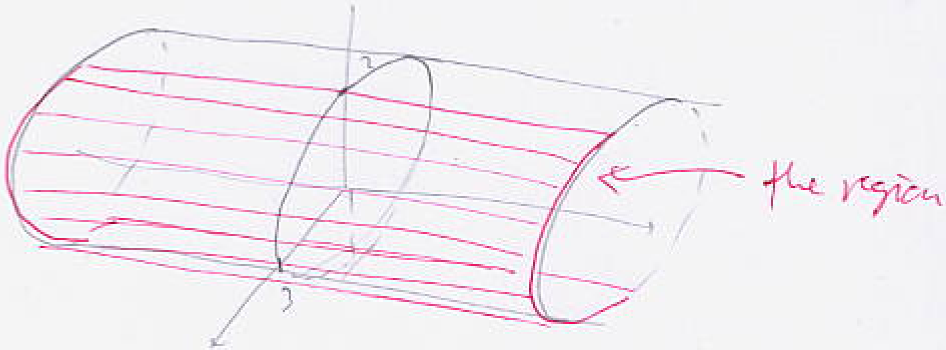
$= 7 \cdot 3 \cdot 1 \cdot 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 63$

3. (8pts) Draw the region in $\mathbf{R}^3$ described by:

$$\frac{x^2}{9} + \frac{z^2}{4} = 1,\ x \geq 0$$



ellipse in xz-plane - so cylinder based on this ellipse



Since $x \geq 0$, only "front" part is included

4. (6pts) Which of the following expressions are meaningful? Briefly explain.

$\mathbf{u} \times (\mathbf{v} \cdot \mathbf{u})$	$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$	$(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v})$
vector × scalar not defined	vector + scalar not defined	scalar · vector is defined

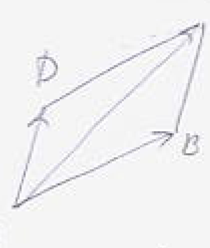
5. (20pts) A parallelogram in $\mathbf{R}^3$ has vertices $A = (3,1,4)$, $B = (5,2,4)$, $C = (8,1,3)$, and $D = (6,0,3)$.

a) Find the equation of the plane that contains this parallelogram.

b) Find the area of the parallelogram.

c) Is this parallelogram a rectangle?

a)



$\vec{AB} = \langle 2, 1, 0 \rangle$

$\vec{AC} = \langle 5, 0, -1 \rangle$

$\vec{AD} = \langle 3, -1, -1 \rangle$

($\vec{AD} + \vec{AB} = \vec{AC}$ so it is a parallelogram)

The plane in question will have normal vector

$$\vec{n} = \vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{vmatrix} = \langle -1, 2, -2-3 \rangle = \langle -1, 2, -5 \rangle$$

Take $\vec{n} = \langle 1, -2, 5 \rangle$

Eq. of plane: $1 \cdot (x-3) - 2 \cdot (y-1) + 5(z-4) = 0$

$x + 2y - 5z = 21$

b) area of parallelogram $= |\vec{AB} \times \vec{AD}| = \sqrt{1^2 + 2^2 + (-5)^2}$

$= \sqrt{30}$

c) $\vec{AB} \cdot \vec{AD} = 6 - 1 + 0 = 5$

so $\vec{AB}$ is not perp to $\vec{AD}$, thus not a rectangle.

($\vec{AB} \cdot \vec{AC} = \langle 2, 1, 0 \rangle \cdot \langle 5, 0, -1 \rangle = 10$ so $\vec{AB}$ is not perp to $\vec{AC}$ either)

6. (22pts) The curve $\mathbf{r}(t) = \langle (t+1)\cos t, (t+1)\sin t, 2t \rangle$ is given, $0 \le t \le 4\pi$.

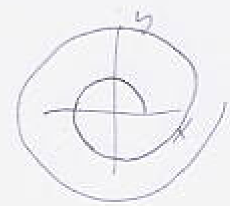
a) Sketch the curve in the coordinate system.

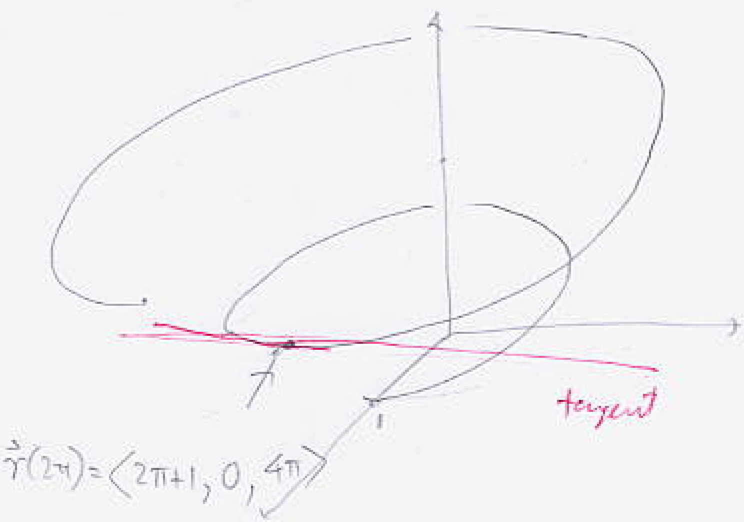
b) Find parametric equations of the tangent line to this curve when $t = 2\pi$ and sketch the tangent line.

c) Set up the integral for the length of the curve. Simplify the function inside the integral as much as possible, but do not evaluate the integral.

a) $\left.\begin{aligned} x &= (t+1)\cos t \\ y &= (t+1)\sin t \end{aligned}\right\}$ rotation in xy plane while radius increase

$z = 2t$ — increasing z at steady rate





b) $\vec{r}\,'(t) = \langle \cos t + (t+1)(-\sin t), \sin t + (t+1)\cos t, 2 \rangle$

$\vec{r}\,'(2\pi) = \langle 1 - (2\pi+1)\cdot 0, 0 + (2\pi+1)\cdot 1, 2 \rangle$

$= \langle 1, 2\pi+1, 2 \rangle$

Eq. of tan line: $x = 2\pi + 1 + t$
$y = (2\pi+1)t$
$z = 4\pi + 2t$

c) $|\vec{r}\,'(t)| = \sqrt{(\cos t - (t+1)\sin t)^2 + (\sin t + (t+1)\cos t)^2 + 2^2}$

$= \sqrt{\cos^2 t - \cancel{2(t+1)\sin t\cos t} + (t+1)^2\sin^2 t + \sin^2 t + \cancel{2(t+1)\sin t\cos t} + (t+1)^2\cos^2 t + 4}$

$= \sqrt{1 + (t+1)^2 + 4} = \sqrt{t^2 + 2t + 6}$

length $= \int_0^{4\pi} \sqrt{t^2+2t+6}\, dt$

7. (16pts) This problem is about the surface $\frac{x^2}{4} - \frac{y^2}{25} - \frac{z^2}{16} = 1$.

a) Identify and sketch the intersections of this surface with the coordinate planes.

b) Sketch the surface in 3D, with coordinate system visible.

a) $x=0$ $\quad -\frac{y^2}{25} - \frac{z^2}{16} = 1$

neg = pos

no sol.

$\Rightarrow$ nothing

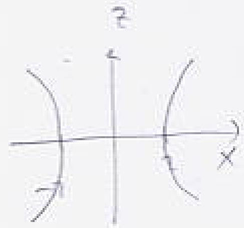
$x = k$

$\frac{k^2}{4} - 1 = \frac{y^2}{25} + \frac{z^2}{16}$

ellipse when $|k|>2$

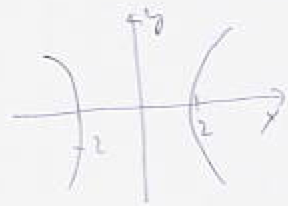
$y=0$ $\quad \frac{x^2}{4} - \frac{z^2}{16} = 1$

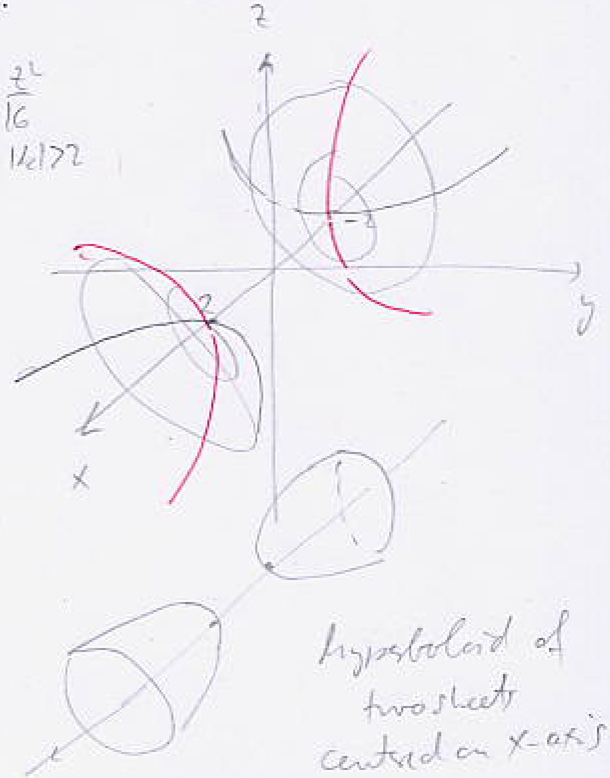
hyperbola



$z=0$ $\quad \frac{x^2}{4} - \frac{y^2}{25} = 1$

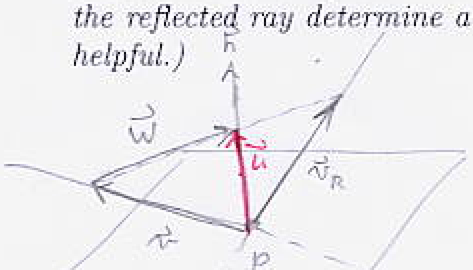
hyperbola





hyperboloid of two sheets centred on x-axis

Bonus (10pts) A ray of light, represented by the line $x = 2 - t$, $y = 4 + 2t$, $z = -3 - 3t$ reflects off the mirror represented by the plane $x - y + 2z = 10$ at point $P = (4, 0, 3)$. Find parametric equations of the line that represents the reflected ray. *(Hints: the ray and the reflected ray determine a plane that is perpendicular to the mirror. Vector projection is helpful.)*



$\vec{v}$ = direction vector of ray

$\vec{v}_R$ = direction vector of reflection

$\vec{n}$ = normal vector of plane.

Let $\vec{u} = \text{proj}_{\vec{n}} \vec{v}$, then $\vec{u} - \vec{v} = \vec{w}$ is perp to $\vec{u}$

$\vec{v}_R = \vec{v} + 2\vec{w}$
$= \vec{v} + 2(\vec{u} - \vec{v})$
$= 2\vec{u} - \vec{v}$

Find $\vec{u} = \text{proj}_{\vec{n}} \vec{v}$

$= \frac{\vec{n} \cdot \vec{v}}{|\vec{n}|^2} \vec{n}$

$= \frac{\langle 1,-1,2\rangle \cdot \langle -1,2,-3\rangle}{1^2 + (-1)^2 + 4} \langle 1,-1,2\rangle$

$-1, -1, \overline{12}$

$= \frac{-9}{6} \langle 1,-1,2\rangle = \langle -\frac{3}{2}, \frac{3}{2}, 3\rangle$

$\vec{v}_R = 2\vec{u} - \vec{v} = 2\langle -\frac{3}{2}, \frac{3}{2}, 3\rangle - \langle -1, 2, -3\rangle$

$= \langle -3+1, 3-2, 6+3\rangle$

$= \langle -2, 1, 9\rangle$

Equation of refl. ray:

$x = 4 - 2t$
$y = t$
$z = 3 + 9t$