Algebra & Trigonometry — Final Exam
MAT 150, Fall 2013 — D. Ivanšić

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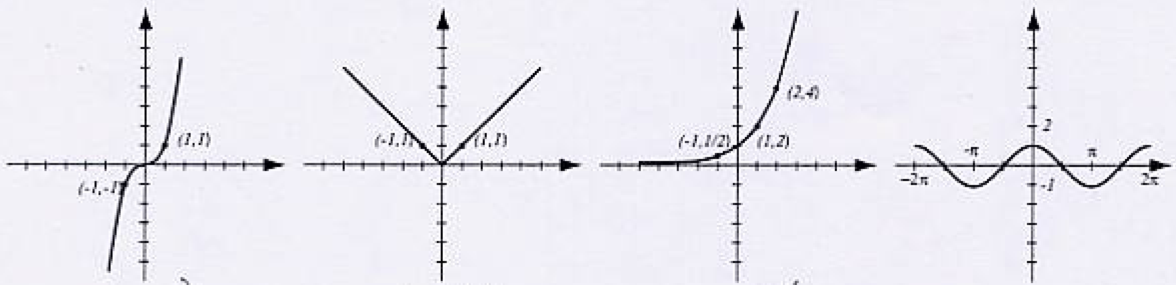
Show all your work!

1. (8pts) The following are graphs of basic functions. Write the equation of the graph under each one.



$y=x^3$ $\qquad$ $y=|x|$ $\qquad$ $y=2^x$ $\qquad$ $y=\cos\theta$

2. (8pts) Find the equation of the line (in form $y = mx + b$) that passes through the points $(1, -2)$ and $(3, 1)$. Then write the equation of the line perpendicular to it that passes through $(1, -2)$. Draw both lines in the coordinate system.

$m = \frac{1-(-2)}{3-1} = \frac{3}{2}$

$y-1 = \frac{3}{2}(x-3)$

$y = \frac{3}{2}x - \frac{9}{2} + 1$

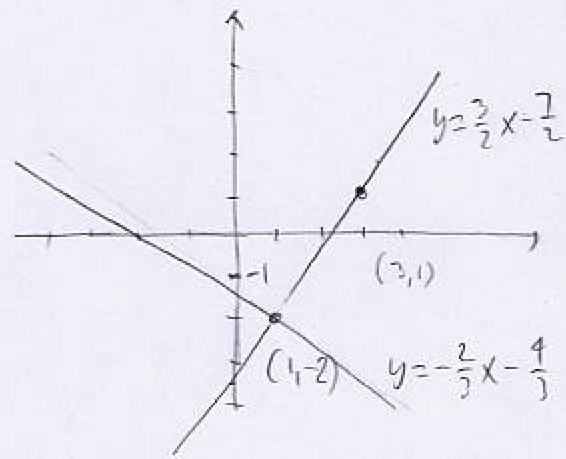
$y = \frac{3}{2}x - \frac{7}{2}$

Perpendicular line has slope $-\frac{1}{\frac{3}{2}} = -\frac{2}{3}x$

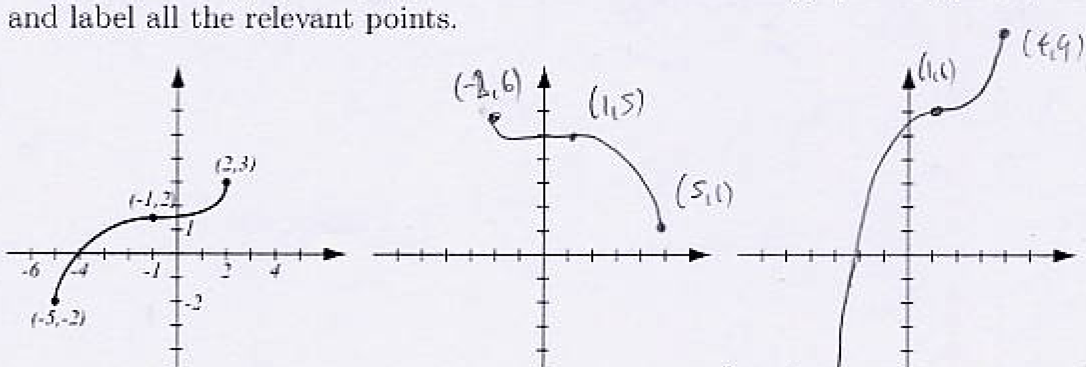
$y-(-2) = -\frac{2}{3}(x-1)$

$y = -\frac{2}{3}x + \frac{2}{3} - 2$

$y = -\frac{2}{3}x - \frac{4}{3}$



3. (10pts) The graph of $f(x)$ is drawn below. Find the graphs of $f(-x) + 3$ and $3f(x - 2)$ and label all the relevant points.



$f(-x)+3$
reflect in y-axis,
shift up 3

$3f(x-2)$
shift right 2
stretch vertically, factor 3

4. (6pts) Find the domain of the function $f(x) = \dfrac{\sqrt{7-x}}{x^2 - 11x + 18}$ and write it in interval notation.

Must have

$7 - x \geq 0$

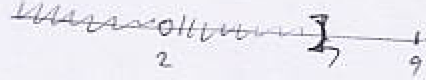
$x \leq 7$

Can't have

$x^2 - 11x + 18 = 0$

$(x-2)(x-9) = 0$

$x = 2, 9$

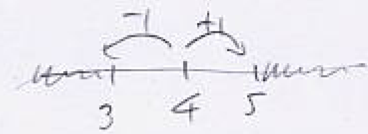


$(-\infty, 2) \cup (2, 7]$

5. (6pts) Solve the inequality. Draw the solution and write it in interval notation.

$|x - 4| \geq 1$

distance from x to $4 \geq 1$



$(-\infty, 3] \cup [5, \infty)$

6. (20pts) Let $f(x) = x^3 - 24x$ (answer with 6 decimal points accuracy).

a) Use your graphing calculator to accurately draw the graph of f (on paper!). Indicate scale on the graph.

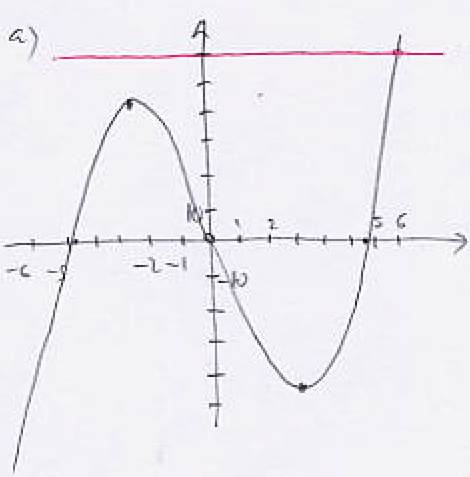
b) Determine algebraically whether f is even, odd, or neither. Then verify your answer by looking at the graph (justify).

c) Find the local maxima and minima for this function.

d) State the intervals where the function is increasing and where it is decreasing.

e) How many solutions does the equation $f(x) = 63$ have? (You do not have to find the solutions.)

a)



b) $f(-x) = (-x)^3 - 24(-x) = -x^3 + 24 = -f(x)$

f is odd, visible from graph being symmetric w.r.t. origin.

c) Local maximum is $f(-2.828428) = 45.25483$

Local minimum is $f(2.828428) = -45.25483$

d) Increasing on $(-\infty, -2.828428) \cup (2.828428, \infty)$

Decreasing on $(-2.828428, 2.828428)$

e) One solution: the line $y = 63$ crosses the graph only once.

7. (14pts) The quadratic function $f(x) = 2x^2 - x - 15$ is given. Do the following without using the calculator.

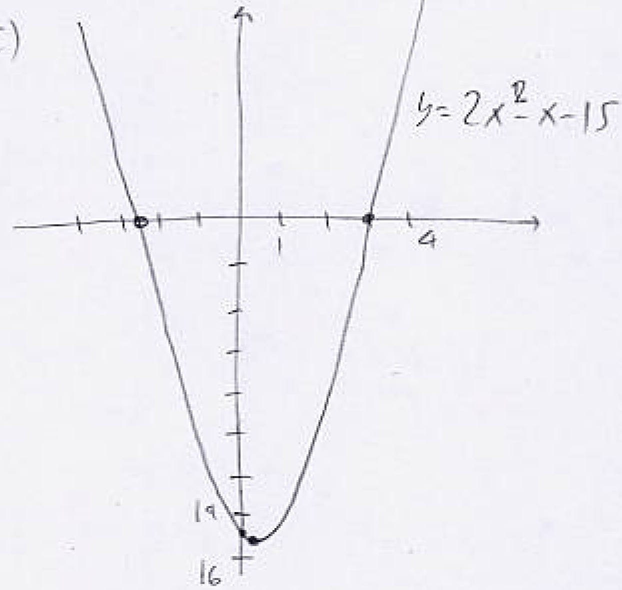
a) Find the x- and y-intercepts of its graph, if any.

b) Find the vertex of the graph.

c) Sketch the graph of the function.

d) What is the range of f?

c)



a) $2x^2 - x - 15 = 0$ $\boxed{\text{y-int, } f(0) = -15}$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-15)}}{2 \cdot 2}$$

$$= \frac{1 \pm \sqrt{121}}{4} = \frac{1 \pm 11}{4} = -\frac{5}{2}, 3 \leftarrow \text{x-int}$$

b) Vertex: $h = -\frac{(-1)}{2 \cdot 2} = \frac{1}{4}$

$$k = 2 \cdot \left(\frac{1}{4}\right)^2 - \frac{1}{4} - 15 = \frac{2}{16} - \frac{1}{4} - 15$$

$$= \frac{1}{8} - 15 = -\frac{121}{8} = -15.125$$

d) Range $= [-\frac{121}{8}, \infty)$

$= [-15.125, \infty)$

8. (6pts) If $\log_a 3 = 0.458157$ and $\log_a 5 = 0.671188$, find (show how you obtained your numbers):

$\log_a 15 = \log_a 3 + \log_a 5$

$= 0.458157 + 0.671188$

$= 1.129345$

$\log_a \frac{9}{125} = \log_a 9 - \log_a 125$

$= \log_a 3^2 - \log_a 5^3$

$= 2\log_a 3 - 3\log_a 5$

$= 2 \cdot 0.458157 - 3 \cdot 0.671188 = -1.09725$

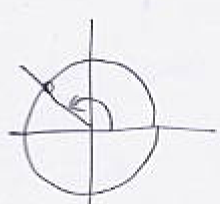
9. (10pts) Without using the calculator, find the exact values of the following trigonometric expressions. Draw the unit circle and the appropriate angle under the expression.

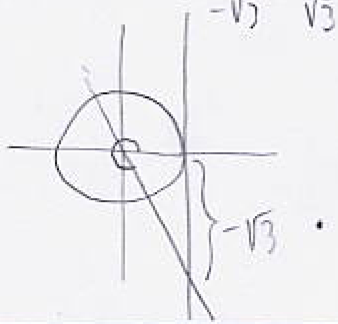
$\cos 135° = -\frac{\sqrt{2}}{2}$

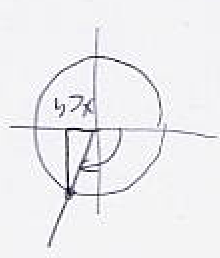
$\cot \frac{5\pi}{3} = \frac{1}{\tan \frac{5\pi}{3}} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$

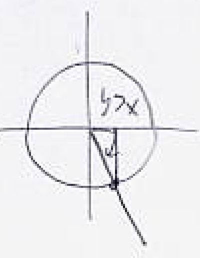
$\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\csc(-60°) = \frac{1}{\sin(-60°)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$

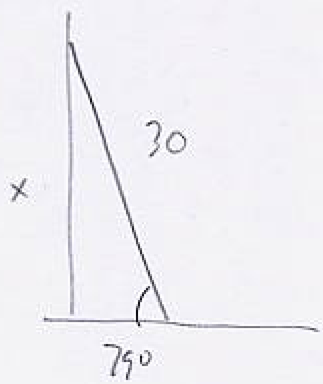








10. (6pts) A 30-ft ladder leans against the wall and makes an angle of 79° with the ground. How high up the wall does the ladder reach?



$$\frac{x}{30} = \sin 79^\circ$$

$$x = 30 \sin 79^\circ$$

$$= 30 \cdot 0.981627$$

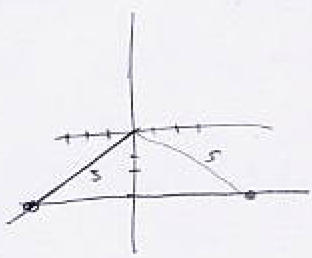
$$= 29.448816$$

11. (10pts) If $\csc\theta = -\frac{5}{3}$ and θ is in the third quadrant, find the other five trigonometric functions of θ. Draw a picture.

$$\csc\theta = -\frac{5}{3}$$

$$\frac{1}{\sin\theta} = -\frac{5}{3}$$

$$\sin\theta = \frac{-3}{5} = \frac{y}{r}$$



$\sin\theta = -\frac{3}{5}$ $\qquad$ $\csc\theta = -\frac{5}{3}$

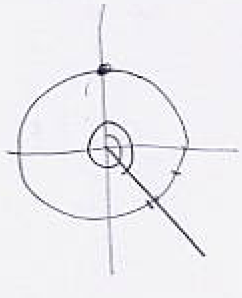
$\cos\theta = -\frac{4}{5}$ $\qquad$ $\sec\theta = -\frac{5}{4}$

$\tan\theta = \frac{-3}{-4} = \frac{3}{4}$ $\qquad$ $\cot\theta = \frac{4}{3}$

$x^2 + (-3) = 5^2$ $\qquad$ $x = \pm 4$

$x^2 = 16$ $\qquad$ $x = -4$ due to 3Q.

12. (6pts) What distance does the tip of the minute hand travel from 4:00PM to 5:24PM if the length of the minute hand is 2 inches?



$$\text{Angle is } 2\pi + \frac{24}{60} \cdot 2\pi = \left(2 + \frac{24}{30}\right)\pi = \left(2 + \frac{4}{5}\right)\pi$$

$$= \frac{14}{5}\pi$$

$$s = r\theta = 2 \cdot \frac{14}{5}\pi = \frac{28}{5}\pi = 17.592919 \text{ in}$$

13. (12pts) According to census data, the population of the Nashville, TN, metro area was about 1,312,000 in 2000 and 1,590,000 in 2010. Assume that the population follows the exponential growth model $P(t) = P_0e^{kt}$.
a) Write the function that describes the population t years since 2000.
b) Graph the function on paper.
c) When will the Nashville metro area reach population 2,000,000?

a) $P = 1312\,e^{kt}$ (in thousands)

Need k

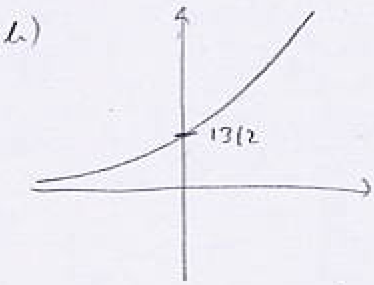
$1590 = 1312\,e^{k\cdot 10}$

$\frac{1590}{1312} = e^{k\cdot 10} \quad |\ \ln$

$\ln\frac{1590}{1312} = 10k$

$k = \frac{\ln\frac{1590}{1312}}{10} = 0.0192181$

b)



c) $2000 = 1312\,e^{0.0192181t} \quad |\div 1312$

$\frac{2000}{1312} = e^{0.019\ldots t} \quad |\ \ln$

$\ln\frac{2000}{1312} = 0.019\ldots t$

$t = \frac{\ln\frac{2000}{1312}}{0.0192181} = 21.937329$ years

About 22 years, so in approx 2022.

14. (14pts) Gina plans to invest \$12,000, part at 4% and the rest at 6% simple interest. What is the most that she can invest at 4% in order to get at least \$650 per year in interest?

x = amount invested at 4%

$12{,}000 - x$ = amount invested at 6%

interest from 4% + interest from 6% ≥ 650

$0.04x + 0.06(12000 - x) \geq 650$

$0.04x + 720 - 0.06x \geq 650 \quad |-720$

$-0.02x \geq -70$

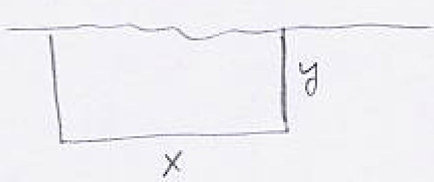
$x \leq \frac{70}{0.02}$

$x \leq 3500$

Can invest at most 3500 at 4% in order to get at least \$650 in interest.

15. (14pts) Farmer Dwayne has 250 meters of fencing that he will use to enclose a rectangular plot of land next to a straight river. The side of the rectangle along the river does not need fencing. Dwayne wishes to maximize the area of the rectangle.
a) Express the area of the enclosure as a function of the length of one of the sides. What is the domain of this function?
b) Sketch the graph the function in order to find the maximum (no need for the graphing calculator — you should already know what the graph looks like). What dimensions of the rectangle give you maximal area?

a)



$x + 2y = 250$

$2y = 250 - x$

$y = 125 - \frac{x}{2}$

$A = xy = x\left(125 - \frac{x}{2}\right)$

$= -\frac{x^2}{2} + 125x$

Domain:

$x \geq 0$

$y \geq 0$

$125 - \frac{x}{2} \geq 0$

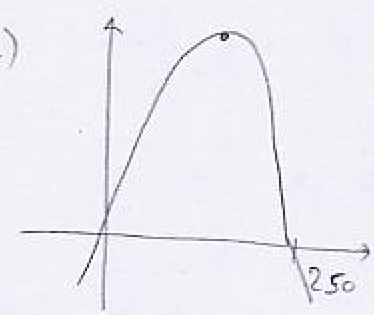
$\frac{x}{2} \leq 125$

$x \leq 250$

Domain

$[0, 250]$.

b)



Max. occurs for

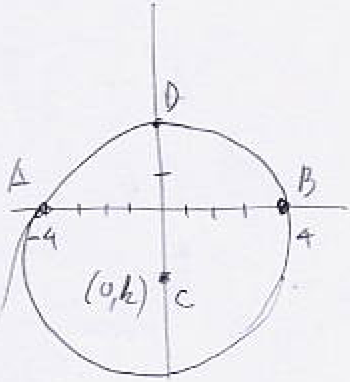
$x = -\frac{125}{2\left(-\frac{1}{2}\right)} = 125$

$y = 125 - \frac{125}{2} = \frac{125}{2} = 62.5$

Biggest area for rectangle:

125×62.5 meters

Bonus (10pts) Find the equation of the circle that passes through points $(-4, 0)$, $(4, 0)$ and $(0, 2)$. *Hint: due to symmetry, the center must be on the y-axis.*



$d(A, C) = d(C, D)$

$\sqrt{(-4-0)^2 + (0-k)^2} = \sqrt{(0-0)^2 + (2-k)^2} \quad |^2$

$16 + k^2 = (2-k)^2$

$16 + \cancel{k^2} = 4 - 4k + \cancel{k^2} \quad | -4$

$-4k = 12$

$k = \frac{12}{-4} = -3$

Center is at $(0, -3)$