Linear Algebra— Exam 1
MAT 335, Spring 2012 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) A^2C b) BB^T c) $2C - B^TA$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 4 & -3 \\ -2 & 0 & -2 \end{bmatrix}$$

2. (8pts) Verify that the following vectors form an orthonormal set (recall this has to do with lengths and dot products):

$$\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right).$$

3. (12pts) Solve both systems below without much computation by using the inverse of the unaugmented matrix of the system.

$$\begin{aligned} 3x_1 \ +5x_2 &= 0 \\ 2x_1 \ +4x_2 &= 1 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 \ +5x_2 &= -5 \\ 2x_1 \ +4x_2 &= 4 \end{aligned}$$

4. (16pts) A system of linear equations is given below.
a) Use Gauss-Jordan elimination (reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.

$$\begin{array}{rrrrl} x_1 & +3x_2 & +3x_3 & -6x_4 & = 14 \\ x_1 & +3x_2 & +4x_3 & -8x_4 & = 18 \\ -x_1 & -3x_2 & -2x_3 & +4x_4 & = -10 \end{array}$$

5. (14pts) A system of linear equations is given below. Use Gaussian elimination (row-echelon form with back substitution) in order to solve the system.

$$\begin{array}{rrrl} -3x_1 & -8x_2 & -13x_3 & = 7 \\ -2x_1 & -5x_2 & -10x_3 & = 5 \\ x_1 & +3x_2 & +3x_3 & = -2 \end{array}$$

6. (6pts) Find a matrix B so that $B\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4c & 4d \\ -a & -b \end{bmatrix}$ for every 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

7. (6pts) The matrix B was obtained by applying a row operation to matrix A. Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 4 & 3 \\ -1 & 5 \\ 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ -1 & 5 \\ -1 & -2 \end{bmatrix} \qquad E =$$

8. (10pts) Below is the augmented matrix of a system of linear equations. Determine the c's for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. Note that no row operations are needed.

$$A = \begin{bmatrix} 1 & -7 & 3 & 5 \\ 0 & c^2 - c - 2 & c+1 & 5c - 10 \\ 0 & 0 & 1 & c+3 \end{bmatrix}$$

9. (16pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{u} \in \mathbf{R}^3$ is orthogonal to all three standard unit vectors $(1,0,0)$, $(0,1,0)$, $(0,1,0)$, then $\mathbf{u} = \mathbf{0}$.
b) An $n \times n$ matrix with at least one zero entry is not invertible.
c) If A is a 3×3 matrix with at least 1 non-zero entry, then the solution set of the linear system $A\mathbf{x} = \mathbf{0}$ has at most 2 parameters.

Bonus. (10pts) Find angle θ so that the matrix A satisfies $A^2 = -I$.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Linear Algebra— Exam 2 **MAT 335, Spring 2012 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

1. (10pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 3 & 5 & -1 & -4 \\ -1 & 4 & 0 & 5 \\ 3 & 6 & -1 & -4 \\ 1 & 1 & 3 & 3 \end{vmatrix} =$$

2. (8pts) Let A, B be 4×4 matrices so that $\det A = 3$ and $\det B = 5$. Compute the following, if possible:

$\det(AB^{-1}) =$

$\det(2B^3) =$

$\det(A+B) =$

3. (8pts) Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose solution is given below (A is a 2×3 matrix).
a) Write any two solutions of the system.
b) Draw the solution set (doesn't have to be accurate, just capture what it looks like).
c) Write the general solution of the system $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 &= 4 \quad -5t \\ x_2 &= -1 \quad +3t \\ x_3 &= 7 \quad -2t \end{aligned}$$

4. (12pts) Determine whether the vectors $(1, 1, 2)$, $(5, 4, -1)$ and $(-7, -5, 8)$ are linearly independent. If they are not, write one as a linear combination of the other two.

5. (12pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$$A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$$

6. (8pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin by $\dfrac{5\pi}{4}$.
a) Write the standard matrix of this transformation.
b) Find $T(-1, 6)$.

7. (14pts) Write the standard matrices for the following linear transformations.
a) $T : \mathbf{R}^3 \to \mathbf{R}^3$, T reflects in the plane spanned by the vectors $\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{e}_2$.
b) $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(\mathbf{e}_1) = (2, 3)$, $T(\mathbf{e}_2) = (-1, 4)$ and $T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (-3, 7)$.

8. (10pts) Show that the set of all vectors of form $(a, b, 0, c)$, where $4a + 3b - 5c = 0$ is a subspace of $\mathbf{R}^4$.

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If B is obtained from an $n \times n$ matrix A by flipping it around a horizontal center line so 1st and n-th, 2nd and $(n-1)$-st, etc. rows exchange places, then $\det B = \det A$.
b) If the 2×2 matrix A has eigenvalues -2 and 5, then A is invertible.
c) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator and $T(\mathbf{x}) = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. (Don't confuse this with the known statement that $T(\mathbf{0}) = \mathbf{0}$.)

Bonus. (10pts) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbf{R}^4$ so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0$. Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. *Hint: definition of linear independence.*

Linear Algebra— Exam 3
MAT 335, Spring 2012 — D. Ivanšić

Name: ______________________

Show all your work!

1. (10pts) Explain which of the following are a basis for the listed space. No computation is needed.
a) $\mathbf{v}_1 = (1, -2, 1)$, $\mathbf{v}_2 = (0, 1, 7)$, $\mathbf{v}_3 = (0, 0, 1)$ for $\mathbf{R}^3$
b) $\mathbf{v}_1 = (-2, 1, 5, 0)$, $\mathbf{v}_2 = (0, 3, 3, 3)$, $\mathbf{v}_3 = (-2, 1, 0, -3)$ for $\mathbf{R}^4$

2. (12pts) Use matrix multiplication to find the matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ that is the composition of a rotation by 90° around the x axis, followed by a projection to the xz-plane.

3. (12pts) Let A be a 4×6 matrix. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) Give an example of a 4×6 matrix whose nullity is 3.

4. (14pts) Find the standard matrix of the linear transformation given by the equations below and determine whether it is a) one-to-one, or b) onto.

$$\begin{aligned} w_1 &= 5x_1 - 3x_2 + x_3 \\ w_2 &= \phantom{5x_1 - 3x_2 + {}} 4x_3 \end{aligned}$$

5. (22pts) A matrix A is given below.
a) Find any basis for row(A).
b) Find any basis for null(A).
c) Find a basis for col(A) that consists of some of the columns of the matrix. Then express the remaining columns as linear combinations of the basis vectors.

$$A = \begin{bmatrix} 1 & 3 & 5 & 5 & 6 \\ 2 & 6 & 9 & 8 & -5 \\ 0 & 0 & 1 & 2 & 17 \end{bmatrix}$$

6. (12pts) Let W be the subspace of $\mathbf{R}^4$ spanned by vectors $(3, 1, 7, 8)$ and $(1, -2, 0, 4)$. Find a basis for $W^{\perp}$.

7. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If the rank of a 4×3 matrix is 3, its columns are linearly independent.

b) If the rank of the augmented matrix $[A \,|\, \mathbf{b}]$ is the same as the rank of the matrix A, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

c) For a linear transformation T_A given by a symmetric matrix A (recall this means $A^T = A$), $\ker T_A = (\text{range } T_A)^{\perp}$.

Bonus. (10pts) Vectors Arrow, Bearing, Course and Direction walk into a 3-dimensional bar.
"I am glad I am not a zero vector. A cousin of mine is — he leads an aimless life," says Arrow.
"If it weren't for me, you girls could not span this bar," says Bearing.
"Well, overBearing, had you not mentioned it, I wouldn't have said anything, but the rest of you cannot span this bar without me, either," says Course.
"I can't say the same for myself, I am afraid, although I am not a zero vector. Since you don't really need me, maybe I'll go home after one drink," says Direction.
All of the vectors spoke the truth. What configuration are they in? Justify your answer. Give an example for such a configuration in coordinates.

Linear Algebra— Final Exam	**Name:** ______________________
MAT 335, Spring 2012 — D. Ivanšić	*Show all your work!*

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) $B^T C$ b) CA c) $AB - BA$

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \end{bmatrix}$$

2. (10pts) Solve both systems below without much computation by using the inverse of the unaugmented matrix of the system.

$$\begin{aligned} 7x_1 \ + 3x_2 &= -3 \\ -x_1 \ + x_2 &= 2 \end{aligned} \qquad \begin{aligned} 7x_1 \ + 3x_2 &= 2 \\ -x_1 \ + x_2 &= 5 \end{aligned}$$

3. (14pts) A system of linear equations is given below.
a) Use Gauss-Jordan elimination (reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.
c) Write the solution of the homogeneous system in vector form. What is the basis of this subspace? What is the dimension?

$$\begin{array}{rrrrl} 3x_1 & +3x_2 & +15x_3 & +15x_4 & = 18 \\ 2x_1 & +3x_2 & +13x_3 & +16x_4 & = 16 \\ 3x_1 & -5x_2 & -9x_3 & -33x_4 & = -14 \\ 3x_1 & +4x_2 & +18x_3 & +21x_4 & = 22 \end{array}$$

4. (12pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 2 & 6 & -4 & -1 & -8 \\ 11 & 3 & 3 & -1 & 7 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 17 & 5 & 3 & 10 \\ 1 & 4 & 2 & 0 & 4 \end{vmatrix} =$$

5. (12pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$$A = \begin{bmatrix} -6 & -3 \\ 3 & 4 \end{bmatrix}$$

6. (14pts) Write the standard matrices for the following linear transformations.
a) $T_1 : \mathbf{R}^3 \to \mathbf{R}^3$, T_1 reflects in the plane spanned by the vectors $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_3$.
b) $T_2 : \mathbf{R}^3 \to \mathbf{R}^3$, T_2 rotates around the y-axis by $120°$ (right-handed rule).
c) The composite $T_2 \circ T_1$.

7. (6pts) Let E_1 be the matrix obtained from I_2 by swapping the two rows and let E_2 be the matrix obtained from I_2 by adding 5 times row 2 to row 1. Find the matrix A if we know

$$E_2E_1A = \begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}$$

8. (20pts) The vectors below span a subspace W.
$\mathbf{v}_1 = (2, 3, -1, 4)$, $\mathbf{v}_2 = (0, 2, -2, 2)$, $\mathbf{v}_3 = (2, 7, -5, 8)$, $\mathbf{v}_4 = (7, 1, 1, 2)$, $\mathbf{v}_5 = (-2, -1, -1, -2)$
a) Find a basis for W that consists of some of the vectors listed. Then express the remaining vectors as linear combinations of the basis vectors.
b) Complete the basis vectors to a basis of $\mathbf{R}^4$ (that is, add some more vectors to the basis to get a basis of $\mathbf{R}^4$, hint: $W^{\perp}$).

9. (16pts) Let A be a 3×4 matrix and T_A its associated linear transformation. Answer the following and justify your answers.
a) What is the biggest $\text{rank}(A)$ could be?
b) What is the smallest $\text{nullity}(A)$ could be?
c) Can T_A ever be onto? If so, give an example.
d) Can T_A ever be one-to-one? If so, give an example.

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If E is an $n \times n$ elementary matrix and A any $n \times n$ matrix, then $\det(EA) = \det(A)$.
b) If A is an invertible $n \times n$ matrix, then for any vector $\mathbf{b} \in \mathbf{R}^n$ the rank of the augmented matrix $[A \,|\, \mathbf{b}]$ is the same as the rank of A.
c) If the 2×2 matrix A has eigenvalues 3 and 7, then A is invertible.
d) If the system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, then the dimensions of the row space of A and of the column space of A are not equal.

11. (10pts) Show that the set of all vectors of form (a, a, b, c), where $7a + b - 5c = 0$ is a subspace of $\mathbf{R}^4$.

Bonus. (8pts) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbf{R}^4$ so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0$. Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. *Hint: definition of linear independence.*

Bonus. (7pts) Find the basis of the subspace described in problem 11.