Linear Algebra— Final Exam MAT 335, Spring 2012 — D. Ivanšić

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1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) B^TC b) CA c) $AB-BA$

$A=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}$

$B=\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}$

$C=\begin{bmatrix}3 & 1 & 0\\ 4 & -1 & -2\end{bmatrix}$

a) $B^TC = \begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1 & 0\\ 4 & -1 & -2\end{bmatrix} = \begin{bmatrix}15 & -2 & -6\\ 5 & -3 & -4\end{bmatrix}$

b) $CA = (2\times 3)(2\times 2)$ is not defined

not same

c) $AB-BA = \begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix} - \begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}$

$= \begin{bmatrix}10 & 5\\ 5 & 5\end{bmatrix} - \begin{bmatrix}2 & 1\\ 1 & 13\end{bmatrix} = \begin{bmatrix}8 & 4\\ 4 & -8\end{bmatrix}$

2. (10pts) Solve both systems below without much computation by using the inverse of the unaugmented matrix of the system.

$$\begin{aligned}7x_1 + 3x_2 &= -3\\ -x_1 + x_2 &= 2\end{aligned} \qquad \begin{aligned}7x_1 + 3x_2 &= 2\\ -x_1 + x_2 &= 5\end{aligned}$$

$A = \begin{bmatrix}7 & 3\\ -1 & 1\end{bmatrix}$ $A^{-1} = \frac{1}{7-(-3)}\begin{bmatrix}1 & -3\\ 1 & 7\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & -3\\ 1 & 7\end{bmatrix} = \begin{bmatrix}\frac{1}{10} & -\frac{3}{10}\\ \frac{1}{10} & \frac{7}{10}\end{bmatrix}$

Solutions:

$\frac{1}{10}\begin{bmatrix}1 & -3\\ 1 & 7\end{bmatrix}\begin{bmatrix}-3\\ 2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}-9\\ 11\end{bmatrix} = \begin{bmatrix}-\frac{9}{10}\\ \frac{11}{10}\end{bmatrix}$

$\frac{1}{10}\begin{bmatrix}1 & -3\\ 1 & 7\end{bmatrix}\begin{bmatrix}2\\ 5\end{bmatrix} = \frac{1}{10}\begin{bmatrix}-13\\ 37\end{bmatrix} = \begin{bmatrix}-\frac{13}{10}\\ \frac{37}{10}\end{bmatrix}$

3. (14pts) A system of linear equations is given below.
a) Use Gauss-Jordan elimination (reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.
c) Write the solution of the homogeneous system in vector form. What is the basis of this subspace? What is the dimension?

$$\begin{aligned} 3x_1 &+3x_2 &+15x_3 &+15x_4 &= 18 \\ 2x_1 &+3x_2 &+13x_3 &+16x_4 &= 16 \\ 3x_1 &-5x_2 &-9x_3 &-33x_4 &= -14 \\ 3x_1 &+4x_2 &+18x_3 &+21x_4 &= 22 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 3 & 3 & 15 & 15 & 18 \\ 2 & 3 & 13 & 16 & 16 \\ 3 & -5 & -9 & -33 & -14 \\ 3 & 4 & 18 & 21 & 22 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{cccc|c} 3 & 3 & 15 & 15 & 18 \\ 2 & 3 & 13 & 16 & 16 \\ 0 & -8 & -24 & -48 & -32 \\ 0 & 1 & 3 & 6 & 4 \end{array}\right] \begin{array}{l} (-1) \\ \\ \text{proportional} \\ \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 2 & 3 & 13 & 16 & 16 \\ 0 & 1 & 3 & 6 & 4 \end{array}\right] (-2) \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 3 & 9 & 18 & 12 \\ 0 & 1 & 3 & 6 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & 3 & 6 & 4 \\ & & s & t & \end{array}\right]$$

proportional

b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-2s+t \\ 4-3s-6t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

c) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

basis vectors for solution subspace

Dimension = 2

4. (12pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 2 & 6 & -4 & -1 & -8 \\ 11 & 3 & 3 & -1 & 7 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 17 & 5 & 3 & 10 \\ 1 & 4 & 2 & 0 & 4 \end{vmatrix} = \left[\text{expand by 3rd row}\right] = -2\begin{vmatrix} 2 & -4 & -1 & -8 \\ 11 & 3 & -1 & 7 \\ 5 & 5 & 3 & 10 \\ 1 & 2 & 0 & 4 \end{vmatrix} \begin{array}{l} (-1) \\ \cdot 3 \end{array} = -2\begin{vmatrix} 2 & -4 & -1 & -8 \\ 9 & 7 & 0 & 15 \\ 11 & -7 & 0 & -14 \\ 1 & 2 & 0 & 4 \end{vmatrix}$$

$$= \left[\text{expand by 3rd colmn}\right] = (-2)(-1)\begin{vmatrix} 9 & 7 & 15 \\ 11 & -7 & -14 \\ 1 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} 9 & 7 & 1 \\ 11 & -7 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \left[\text{expand by 3rd colmn}\right]$$

$\cdot(-2)$

$$= 2\begin{vmatrix} 11 & -7 \\ 1 & 2 \end{vmatrix} = 2(22-(-7)) = 58$$

5. (12pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$$A = \begin{bmatrix} -6 & -3 \\ 3 & 4 \end{bmatrix}$$

a) $\det(\lambda I - A) = \begin{vmatrix} \lambda+6 & 3 \\ -3 & \lambda-4 \end{vmatrix}$

$= (\lambda+6)(\lambda-4) - (-9)$

$= \lambda^2 + 2\lambda - 24 + 9$

$= \lambda^2 + 2\lambda - 15$

$= (\lambda+5)(\lambda-3)$

Eigenvalues: $-5, 3$

b) $-5I - A = \begin{bmatrix} 1 & 3 \\ -3 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$

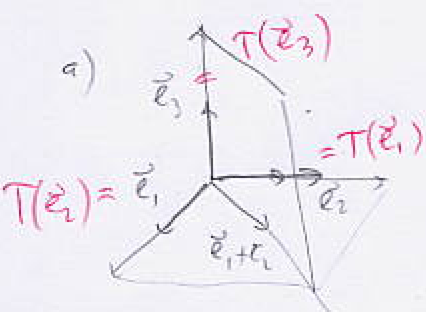
$= t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ eigenvector for $\lambda = -5$

$3I - A = \begin{bmatrix} 9 & 3 \\ -3 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ eigenvector for $\lambda = 3$

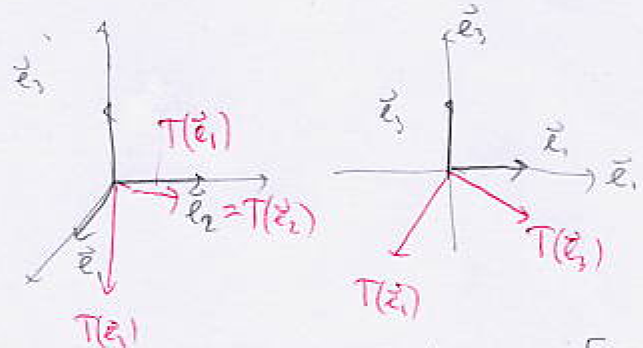
6. (14pts) Write the standard matrices for the following linear transformations.
a) $T_1 : \mathbf{R}^3 \to \mathbf{R}^3$, T_1 reflects in the plane spanned by the vectors $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_3$.
b) $T_2 : \mathbf{R}^3 \to \mathbf{R}^3$, T_2 rotates around the y-axis by $120°$ (right-handed rule).
c) The composite $T_2 \circ T_1$.

a)



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)



In the xz-plane, it is a rotation by $-120°$

$$\begin{bmatrix} \cos(-120°) & 0 & -\sin(-120°) \\ 0 & 1 & 0 \\ \sin(-120°) & 0 & \cos(-120°) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

c) $$T_2 \circ T_1 = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

7. (6pts) Let E_1 be the matrix obtained from I_2 by swapping the two rows and let E_2 be the matrix obtained from I_2 by adding 5 times row 2 to row 1. Find the matrix A if we know

$$E_2E_1A = \begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}$$

$$A = (E_2E_1)^{-1}\begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix} = \underset{\text{swaps rows}}{E_1^{-1}}\ \underset{\text{subtracts } 5\cdot\text{row 2 from row 1}}{E_2^{-1}}\begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix} = E_1^{-1}\begin{bmatrix} -13 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -13 & 3 \end{bmatrix}$$

8. (20pts) The vectors below span a subspace W.
$\mathbf{v}_1 = (2,3,-1,4)$, $\mathbf{v}_2 = (0,2,-2,2)$, $\mathbf{v}_3 = (2,7,-5,8)$, $\mathbf{v}_4 = (7,1,1,2)$, $\mathbf{v}_5 = (-2,-1,-1,-2)$
a) Find a basis for W that consists of some of the vectors listed. Then express the remaining vectors as linear combinations of the basis vectors.
b) Complete the basis vectors to a basis of $\mathbf{R}^4$ (that is, add some more vectors to the basis to get a basis of $\mathbf{R}^4$, hint: $W^\perp$).

a) put in 1st row

$$\begin{bmatrix} 2 & 0 & 2 & 7 & -2 \\ 3 & 2 & 7 & 1 & -1 \\ -1 & -2 & -5 & 1 & -1 \\ 4 & 2 & 8 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 & -1 & 1 \\ 2 & 0 & 2 & 7 & -2 \\ 3 & 2 & 7 & 1 & -1 \\ 4 & 2 & 8 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 & -1 & 1 \\ 0 & -4 & -8 & 9 & -4 \\ 0 & -4 & -8 & 4 & -4 \\ 0 & -6 & -12 & 8 & -6 \end{bmatrix} \leftarrow \text{divide by } -4$$

$$\to \begin{bmatrix} 1 & 2 & 5 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & -4 & -8 & 9 & -4 \\ 0 & -6 & -12 & 8 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & -1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{5} & \cancel{0} \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \leftarrow \div 2 \to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

pivot colums

$\vec{v}_1, \vec{v}_2, \vec{v}_4$ are a basis $\quad \vec{v}_3 = \vec{v}_1 + 2\vec{v}_2,\ \vec{v}_5 = -\vec{v}_1 + \vec{v}_2$

b) Find $\perp$ of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_4$

$$\begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & 2 & -2 & 2 \\ 7 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & -8 & 4 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -8 & 4 & -10 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -8 & 4 & -10 \\ 0 & 1 & -1 & 1 \\ 0 & 19 & -9 & 24 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 10 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} t$$

Add $(0, -\frac{3}{2}, -\frac{1}{2}, 1)$ to $\vec{v}_1, \vec{v}_2, \vec{v}_4$ to get basis.

9. (16pts) Let A be a 3×4 matrix and T_A its associated linear transformation. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) Can T_A ever be onto? If so, give an example.
d) Can T_A ever be one-to-one? If so, give an example.

a) rank $A \leq$ smaller of 3, 4

rank $A \leq 3$

b) nullity $A = 4 -$ rank $A \geq 4-3$

nullity ≥ 1

c) T_A is onto when rank $A = 3$

Ex: $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

d) T_A is one-to-one iff nullity $A = 0$

Since nullity$(A) \geq 1$, this is not possible.

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If E is an $n \times n$ elementary matrix and A any $n \times n$ matrix, then $\det(EA) = \det(A)$.
b) If A is an invertible $n \times n$ matrix, then for any vector $\mathbf{b} \in \mathbf{R}^n$ the rank of the augmented matrix $[A \,|\, \mathbf{b}]$ is the same as the rank of A.
c) If the 2×2 matrix A has eigenvalues 3 and 7, then A is invertible.
d) If the system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, then the dimensions of the row space of A and of the column space of A are not equal.

a) False. Multiplication by E may swap rows, which changes sign of determinant.

Ex: $A = I$ $\quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \left.\begin{matrix} \det EI = \det E = -1 \\ \det I = 1 \end{matrix}\right\}$ not equal

b) True. If A is invertible, then the system $A\vec{x} = \vec{b}$ is consistent,

so rank $[A \,|\, \vec{b}] =$ rank $[A]$

c) True. If A has eigenvalues 3 and 7, then $p(\lambda) = (\lambda-3)(\lambda-7)$

Since $\det(A) = p(0) = 21$, then $\det A \neq 0$ so A is invertible

d) False. Row space and column space of A always have equal dimension.

Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ yet dim row$(A) =$ dim col$(A) = 2$

$A\vec{x} = \vec{0}$ has nontrivial solution $\vec{x} = \vec{e}_3$

11. (10pts) Show that the set of all vectors of form (a, a, b, c), where $7a + b - 5c = 0$ is a subspace of $\mathbf{R}^4$.

Let $\vec{u} = (a_1, a_1, b_1, c_1)$, $\vec{v} = (a_2, a_2, b_2, c_2)$ satisfy $7a_1 + b_1 - 5c_1 = 0$, $7a_2 + b_2 - 5c_2 = 0$

Then $\vec{u} + \vec{v} = (a_1 + a_2, a_1 + a_2, b_1 + b_2, c_1 + c_2) \leftarrow$ has same form

$$7(a_1 + a_2) + (b_1 + b_2) - 5(c_1 + c_2) = 7a_1 + b_1 - 5c_1 + 7a_2 + b_2 - 5c_2 = 0 + 0 = 0$$

$t\vec{u} = (ta_1, ta_1, tb_1, tc_1)$, $\quad 7ta_1 + tb_1 - 5tc_1 = t(7a_1 + b_1 - 5c_1) = t0 = 0$

Since $\vec{u} + \vec{v}$ and $t\vec{u}$ have the same form and satisfy the equation, we conclude the set is closed under addition and multiplication by scalar, hence is a subspace.

Bonus. (8pts) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbf{R}^4$ so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0$. Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. *Hint: definition of linear independence.*

Let $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0} \quad | \cdot \vec{u}$,

$$c_1(\vec{u} \cdot \vec{u}) + c_2\underbrace{(\vec{v} \cdot \vec{u})}_{=0} + c_3\underbrace{(\vec{w} \cdot \vec{u})}_{=0} = \underbrace{\vec{0} \cdot \vec{u}}_{=0}$$

$$c_1 \|\vec{u}\|^2 = 0$$

since $\|\vec{u}\|^2 \neq 0$, we conclude $c_1 \neq 0$.

Similarly, we get $c_2 = c_3 = 0$

Thus, c_1, c_2, c_3 are necessarily 0, hence the vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

Bonus. (7pts) Find the basis of the subspace described in problem 11.

$7a + b - 5c = 0$ so $b = 5c - 7a$

$(a, a, 5c - 7a, c) = (a, a, -7a, 0) + (0, 0, 5c, c) = a(1, 1, -7, 0) + c(0, 0, 5, 1)$

Thus, $(1, 1, -7, 0)$ and $(0, 0, 5, 1)$ span the subspace. Since they are not proportional, they are linearly independent, hence form a basis for the subspace.