Linear Algebra— Exam 3
MAT 335, Spring 2012 — D. Ivanšić

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Show all your work!

1. (10pts) Explain which of the following are a basis for the listed space. No computation is needed.
a) $\mathbf{v}_1 = (1, -2, 1)$, $\mathbf{v}_2 = (0, 1, 7)$, $\mathbf{v}_3 = (0, 0, 1)$ for $\mathbf{R}^3$
b) $\mathbf{v}_1 = (-2, 1, 5, 0)$, $\mathbf{v}_2 = (0, 3, 3, 3)$, $\mathbf{v}_3 = (-2, 1, 0, -3)$ for $\mathbf{R}^4$

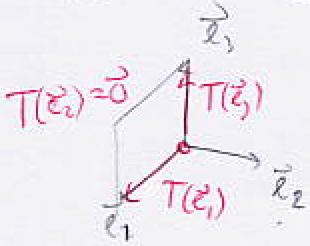
a) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ is in RE-form, so rows are linearly independent

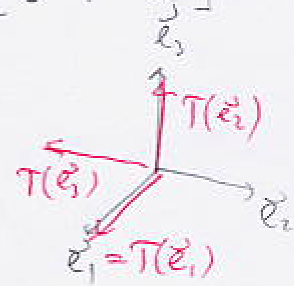
Since there are three of them, they span $\mathbb{R}^3$, hence are a basis

b) Three vectors cannot span $\mathbb{R}^4$ — we need at least 4.

2. (12pts) Use matrix multiplication to find the matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ that is the composition of a rotation by $90°$ around the x axis, followed by a projection to the xz-plane.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$





3. (12pts) Let A be a 4×6 matrix. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) Give an example of a 4×6 matrix whose nullity is 3.

a) $\operatorname{rank} A \leq \min\{4, 6\}$

$\operatorname{rank} A \leq 4$

b) $\operatorname{nullity} A + \operatorname{rank} A = 6$

$\operatorname{nullity} A = 6 - \operatorname{rank} A \geq 6 - 4$

$\operatorname{nullity} A \geq 2$

c) $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. (14pts) Find the standard matrix of the linear transformation given by the equations below and determine whether it is a) one-to-one, or b) onto.

$$\begin{aligned} w_1 &= 5x_1 - 3x_2 + x_3 \\ w_2 &= \qquad\qquad\quad 4x_3 \end{aligned}$$

$$A = \begin{bmatrix} 5 & -3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Since $\ker A = \text{null}A \neq \{\vec{0}\}$, T_A is not one-to-one $\left(\text{null}A = t\begin{bmatrix} \frac{3}{5} \\ 1 \\ 0 \end{bmatrix}\right)$

b) Since $\text{rank}A = 2 =$ dimension of codomain, T_A is onto

5. (22pts) A matrix A is given below.
a) Find any basis for row(A).
b) Find any basis for null(A).
c) Find a basis for col(A) that consists of some of the columns of the matrix. Then express the remaining columns as linear combinations of the basis vectors.

$$A = \begin{bmatrix} 1 & 3 & 5 & 5 & 6 \\ 2 & 6 & 9 & 8 & -5 \\ 0 & 0 & 1 & 2 & 17 \end{bmatrix} \xrightarrow{\cdot(-2)} \begin{bmatrix} 1 & 3 & 5 & 5 & 6 \\ 0 & 0 & -1 & -2 & -17 \\ 0 & 0 & 1 & 2 & 17 \end{bmatrix} \xrightarrow{\cdot(-5)} \begin{array}{ccccc} & & t & s & w \end{array}\begin{bmatrix} 1 & 3 & 0 & -5 & -79 \\ 0 & 0 & 1 & 2 & 17 \end{bmatrix}$$

a) Basis for row $A = \{(1,3,0,-5,-79), (0,0,1,2,17)\}$

b) Basis for null $A = \{(-3,1,0,0,0), (5,0,-2,1,0),$
$(79,0,-17,0,1)\}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 79 \\ 0 \\ -17 \\ 0 \\ 1 \end{bmatrix}$$

c) Columns corresponding to pivot columns in RRE form are the basis: $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 1 \end{bmatrix}\right\} = \{\vec{c}_1, \vec{c}_3\}$

$\vec{c}_2 = 3\vec{c}_1$, $\vec{c}_4 = 5\vec{c}_1 + 2\vec{c}_2$, $\vec{c}_5 = -79\vec{c}_1 + 17\vec{c}_3$

6. (12pts) Let W be the subspace of $\mathbf{R}^4$ spanned by vectors $(3,1,7,8)$ and $(1,-2,0,4)$. Find a basis for $W^\perp$.

Need to find all $\vec{x}$ s.t. $(3,1,7,8)\cdot\vec{x}=0$
$(1,-2,0,4)\cdot\vec{x}=0$

which is same as solving homogeneous system with matrix

$$\begin{bmatrix} 3 & 1 & 7 & 8 \\ 1 & -2 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 7 & 8 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 7 & -4 \end{bmatrix} \xrightarrow{/7} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -\frac{4}{7} \end{bmatrix} \cdot 2$$

$$\to \begin{bmatrix} 1 & 0 & 2 & \frac{20}{7} \\ 0 & 1 & 1 & -\frac{4}{7} \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{20}{7} \\ +\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$$

$s \quad t$

$\leftarrow$ these vectors are a basis for $W^\perp$

7. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If the rank of a 4×3 matrix is 3, its columns are linearly independent.

b) If the rank of the augmented matrix $[A\,|\,\mathbf{b}]$ is the same as the rank of the matrix A, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

c) For a linear transformation T_A given by a symmetric matrix A (recall this means $A^T = A$), $\ker T_A = (\text{range } T_A)^\perp$.

a) True. If rank $A = 3$, nullity $= 3-3=0$, hence $A\vec{x}=\vec{0}$ only has the trivial solution. If $\vec{c}_1, \vec{c}_2, \vec{c}_3$ are columns of A, this means that $x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 = \vec{0}$ can only be satisfied with $x_1 = x_2 = x_3 = 0$, i.e. the columns are lin. independent.

b) False: Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\vec{x}=\vec{b}$ has the unique solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ while rank $[A|\vec{b}] = 2 = $ rank A

c) True: $\ker T_A = \text{null} A = (\text{row} A)^\perp \underset{\uparrow}{=} (\text{col} A)^\perp = (\text{range} A)^\perp$

since $A = A^T$

$\vec{a}$ $\vec{b}$ $\vec{c}$ $\vec{d}$

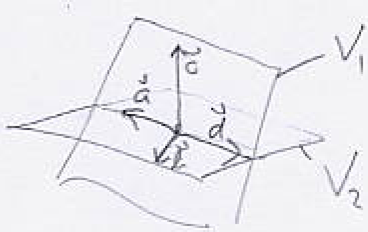
Bonus. (10pts) Vectors Arrow, Bearing, Course and Direction walk into a 3-dimensional bar.
"I am glad I am not a zero vector. A cousin of mine is — he leads an aimless life," says Arrow.
"If it weren't for me, you girls could not span this bar," says Bearing.
"Well, overBearing, had you not mentioned it, I wouldn't have said anything, but the rest of you cannot span this bar without me, either," says Course.
"I can't say the same for myself, I am afraid, although I am not a zero vector. Since you don't really need me, maybe I'll go home after one drink," says Direction.
All of the vectors spoke the truth. What configuration are they in? Justify your answer. Give an example for such a configuration in coordinates.

Since $\{\vec{a}, \vec{c}, \vec{d}\}$ do not span $\mathbb{R}^3$, they must all lie in a plane V_1

$\{\vec{a}, \vec{b}, \vec{d}\}$ " " " " V_2

If $V_1 = V_2$, the vectors $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ would not span $\mathbb{R}^3$, so $V_1 \neq V_2$

Hence $V_1 \cap V_2$ is a line that contains $\vec{a}, \vec{d}$. Thus, $\vec{d} = k\vec{a}$.



Example: $\vec{a} = (1, 0, 0)$
$\vec{b} = (0, 1, 0)$
$\vec{c} = (0, 0, 1)$
$\vec{d} = (-2, 0, 0)$