Linear Algebra— Exam 2 **MAT 335, Spring 2012 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (10pts) Evaluate the determinant by any (efficient) method:

$$\cdot(-1)\quad \begin{vmatrix}3&5&-1&-4\\-1&4&0&5\\3&6&-1&-4\\1&1&3&3\end{vmatrix} = \begin{vmatrix}3&5&-1&-4\\-1&4&0&5\\0&1&0&0\\1&1&3&3\end{vmatrix} = \begin{matrix}\text{expand}\\\text{by}\\\text{3rd row}\end{matrix} = (-1)\begin{vmatrix}3&-1&-4\\-1&0&5\\1&3&3\end{vmatrix}$$

$$= \begin{pmatrix}\text{expand by}\\\text{2nd row}\end{pmatrix} = (-1)\left(\underbrace{(-1)(-1)}_{=1}\begin{vmatrix}-1&-4\\3&3\end{vmatrix} - 5\begin{vmatrix}3&-1\\1&3\end{vmatrix}\right)$$

$$= -\left(-3+12-5(9+1)\right) = -(9-50) = 41$$

2. (8pts) Let A, B be 4×4 matrices so that $\det A = 3$ and $\det B = 5$. Compute the following, if possible:

$\det(AB^{-1}) =$ $\det A \det B^{-1} = \det A \frac{1}{\det B} = 3 \cdot \frac{1}{5} = \frac{3}{5}$

$\det(2B^3) =$ $2^4 \det B^3 = 2^4 (\det B)^3 = 2^4 5^3 = 16 \cdot 125 = 2000$

$\det(A+B) =$ not possible to determine (not $\det A + \det B$)

3. (8pts) Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose solution is given below (A is a 2×3 matrix).
a) Write any two solutions of the system.
b) Draw the solution set (doesn't have to be accurate, just capture what it looks like).
c) Write the general solution of the system $A\mathbf{x} = \mathbf{0}$.

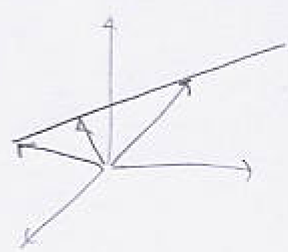
$$\begin{aligned} x_1 &= 4 - 5t \\ x_2 &= -1 + 3t \\ x_3 &= 7 - 2t \end{aligned}$$

a) Let

$t=0 \quad \vec{x} = \begin{bmatrix}4\\-1\\7\end{bmatrix}$

$t=1 \quad \vec{x} = \begin{bmatrix}-1\\2\\5\end{bmatrix}$

b)



All vectors whose tips are on a line

c) $x_1 = -5t$, $x_2 = 3t$, $x_3 = -2t$ or $\vec{x} = t\begin{bmatrix}-5\\3\\-2\end{bmatrix}$

4. (12pts) Determine whether the vectors $(1, 1, 2)$, $(5, 4, -1)$ and $(-7, -5, 8)$ are linearly independent. If they are not, write one as a linear combination of the other two.

Check if $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ only has the trivial solution

$$c_1\begin{bmatrix}1\\1\\2\end{bmatrix} + c_2\begin{bmatrix}5\\4\\-1\end{bmatrix} + c_3\begin{bmatrix}-7\\-5\\8\end{bmatrix} = \vec{0} \iff \begin{bmatrix}1 & 5 & -7\\1 & 4 & -5\\2 & -1 & 8\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \vec{0}$$

Solve homogeneous system

$$\begin{bmatrix}1 & 5 & -7\\1 & 4 & -5\\2 & -1 & 8\end{bmatrix} \xrightarrow{\cdot(-1),\ \cdot(-1)} \begin{bmatrix}1 & 5 & -7\\0 & -1 & 2\\0 & -11 & 22\end{bmatrix} \xrightarrow{\cdot 5,\ \cdot(-11)} \begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

t

$c_3 = t$
$c_1 = -3t$
$c_2 = 2t$

Let $t_1 = 1$, we have $c_3 = 1$, $c_1 = -3$, $c_2 = 2$

$-3v_1 + 2\vec{v}_2 + \vec{v}_3 = 0$

$\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2$

5. (12pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$$A = \begin{bmatrix}3 & 2\\7 & -2\end{bmatrix}$$

a) $\det(\lambda I - A) = \begin{vmatrix}\lambda-3 & -2\\-7 & \lambda+2\end{vmatrix}$

$= (\lambda-3)(\lambda+2) - 14$
$= \lambda^2 - \lambda - 20$

$\lambda^2 - \lambda - 20 = 0$
$(\lambda-5)(\lambda+4) = 0$
$\lambda = 5, -4$

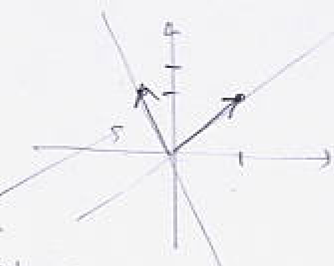
b) 1) $\lambda = 5$ $\quad \begin{bmatrix}2 & -2\\-7 & 7\end{bmatrix} \to \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ $\quad x_2 = t$, $x_1 = t$

$\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix}$

2) $\lambda = -4$ $\quad \begin{bmatrix}-7 & -2\\-7 & -2\end{bmatrix} \to \begin{bmatrix}7 & 2\\0 & 0\end{bmatrix} \to \begin{bmatrix}1 & \frac{2}{7}\\0 & 0\end{bmatrix}$

$x_2 = t$, $x_1 = -\frac{2}{7}t$

$\vec{x} = t\begin{bmatrix}-2/7\\1\end{bmatrix}$

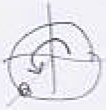


Span eigenspaces

6. (8pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin by $\frac{5\pi}{4}$.

a) Write the standard matrix of this transformation.

b) Find $T(-1, 6)$.



$$[T] = \begin{bmatrix} \cos\frac{5\pi}{4} & -\sin\frac{5\pi}{4} \\ \sin\frac{5\pi}{4} & \cos\frac{5\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

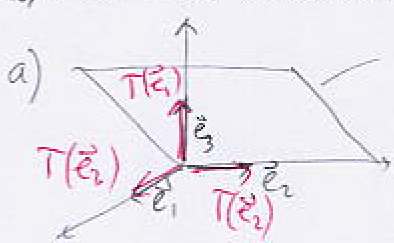
$$T(-1,6) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{5\sqrt{2}}{2} \end{bmatrix}$$

7. (14pts) Write the standard matrices for the following linear transformations.

a) $T : \mathbf{R}^3 \to \mathbf{R}^3$, T reflects in the plane spanned by the vectors $\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{e}_2$.

b) $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(\mathbf{e}_1) = (2, 3)$, $T(\mathbf{e}_2) = (-1, 4)$ and $T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (-3, 7)$.

a)



spaned by $\vec{e}_1+\vec{e}_3$, $\vec{e}_2$

$$[T] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b) $T(\vec{e}_1+\vec{e}_2+\vec{e}_3) = T(\vec{e}_1)+T(\vec{e}_2)+T(\vec{e}_3)$

$(2,3) + (-1,4) + T(\vec{e}_3) = (-3,7)$

$T(\vec{e}_3) = (-3,7)-(2,3)-(-1,4) = (-4,0)$

$$[T] = \begin{bmatrix} 2 & -1 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

2×3

8. (10pts) Show that the set of all vectors of form $(a, b, 0, c)$, where $4a + 3b - 5c = 0$ is a subspace of $\mathbf{R}^4$.

Let $\underbrace{(a_1, b_1, 0, c_1)}_{\vec{u}}$, $\underbrace{(a_2, b_2, 0, c_2)}_{\vec{v}}$ be vectors satisfying $4a_1+3b_1-5c_1=0$, $4a_2+3b_2-5c_2=0$

$\vec{u}+\vec{v} = (a_1+a_2, b_1+b_2, 0, c_1+c_2)$

$$4(a_1+a_2)+3(b_1+b_2)-5(c_1+c_2) = \underbrace{4a_1+3b_1-5c_1}_{=0} + \underbrace{4a_2+3b_2-5c_2}_{=0} = 0$$

Coordinates of $\vec{u}+\vec{v}$ satisfy same equation and have 3rd coord 0, so $\vec{u}+\vec{v}$ is in subset. Same for $t\vec{u}$.

$t\vec{u} = (ta_1, tb_1, 0, tc_1)$ $\quad 4ta_1+3tb_1-5tc_1 = t(\underbrace{4a_1+3b_1-5c_1}_{=0}) = 0$

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If B is obtained from an $n \times n$ matrix A by flipping it around a horizontal center line so 1st and n-th, 2nd and $(n-1)$-st, etc. rows exchange places, then $\det B = \det A$.
b) If the 2×2 matrix A has eigenvalues -2 and 5, then A is invertible.
c) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator and $T(\mathbf{x}) = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. (Don't confuse this with the known statement that $T(\mathbf{0}) = \mathbf{0}$.)

a) False. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det A = 1$ $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\det B = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$
$= 1 \cdot (-1) = -1$

b) $\det(\lambda I - A) = \lambda^2 + p\lambda + q$ a quadratic polynomial
Since $\lambda^2 + p\lambda + q = 0$ has two solutions $-2, 5$ it cannot have solution 0. Hence 0 is not an eigenvalue, so A is invertible.
OR: $\det A =$ product of eigenvalues $= (-2)5 = -10 \neq 0$, so A is invertible.

c) False. Let $T(\vec{x}) = A\vec{x}$. We know from examples that if $A\vec{x} = \vec{0}$, $\vec{x}$ may not be trivial.

For example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A\vec{e}_2 = \vec{0}$, yet $\vec{e}_2 \neq \vec{0}$

Bonus. (10pts) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbf{R}^4$ so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0$. Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. *Hint: definition of linear independence.*

Let c_1, c_2, c_3 be s.t.
$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ $\quad | \cdot \vec{u}, \cdot \vec{v}, \cdot \vec{w}$

$c_1(\vec{u}\cdot\vec{u}) + c_2 \cdot 0 + c_3 \cdot 0 = 0$
$c_1 \cdot 0 + c_2(\vec{v}\cdot\vec{v}) + c_3 \cdot 0 = 0$
$c_1 \cdot 0 + c_2 \cdot 0 + c_3(\vec{w}\cdot\vec{w}) = 0$

Since $\vec{u}\cdot\vec{u}$, $\vec{v}\cdot\vec{v}$, $\vec{w}\cdot\vec{w}$ are all nonzero, it follows that
$c_1 = c_2 = c_3 = 0$
so $\vec{u}, \vec{v}, \vec{w}$ are lin. independent