Linear Algebra— Exam 1 **MAT 335, Spring 2012 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) A^2C b) BB^T c) $2C - B^TA$

$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$C = \begin{bmatrix} 7 & 4 & -3 \\ -2 & 0 & -2 \end{bmatrix}$

a) $A^2 = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$

$A^2C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 7 & 4 & -3 \\ -2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 19 & 12 & -11 \\ -7 & -4 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & 3 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}$

c) B^TA is $(1\times 2)(2\times 3) = 1\times 2$

$2C$ is 2×3 so $2C$ and B^TA can't be added

2. (8pts) Verify that the following vectors form an orthonormal set (recall this has to do with lengths and dot products):

$$\underbrace{\left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)}_{\vec{u}_1}, \underbrace{\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 0\right)}_{\vec{u}_2}, \underbrace{\left(0, \frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right)}_{\vec{u}_3}.$$

$\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{4} - \frac{1}{4} = 0$

$\vec{u}_1 \cdot \vec{u}_3 = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$

$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$

$\|\vec{u}_1\| = \sqrt{\frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2}} = \sqrt{1} = 1$

$\|\vec{u}_2\| = \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1$

$\|\vec{u}_3\| = \sqrt{0 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}} = \sqrt{1} = 1$

3. (12pts) Solve both systems below without much computation by using the inverse of the unaugmented matrix of the system.

$$\begin{aligned} 3x_1 + 5x_2 &= 0 \\ 2x_1 + 4x_2 &= 1 \end{aligned} \qquad \begin{aligned} 3x_1 + 5x_2 &= -5 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{12-10}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$

Solution for 1st eq.: $\begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 3/2 \end{bmatrix}$

2nd eq.: $\begin{bmatrix} 2 & -\frac{5}{2} \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -10+10 \\ -5+6 \end{bmatrix} = \begin{bmatrix} -20 \\ 11 \end{bmatrix}$

4. (16pts) A system of linear equations is given below.
a) Use Gauss-Jordan elimination (reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.

$$\begin{array}{rrrrl} x_1 & +3x_2 & +3x_3 & -6x_4 & = 14 \\ x_1 & +3x_2 & +4x_3 & -8x_4 & = 18 \\ -x_1 & -3x_2 & -2x_3 & +4x_4 & = -10 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & -6 & 14 \\ 1 & 3 & 4 & -8 & 18 \\ -1 & -3 & -2 & 4 & -10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & -6 & 14 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -2 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & -6 & 14 \\ 0 & 0 & 1 & -2 & 4 \end{array}\right] \cdot(-3)$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 & 4 \end{array}\right]$$

$x_1 + 3x_2 = 2$

$x_3 - 2x_4 = 4$

$x_2 = s, \; x_4 = t$

$x_1 = 2 - 3s$

$x_3 = 4 + 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-3s \\ s \\ 4+2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

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5. (14pts) A system of linear equations is given below. Use Gaussian elimination (row-echelon form with back substitution) in order to solve the system.

$$\begin{array}{rrrl} -3x_1 & -8x_2 & -13x_3 & = 7 \\ -2x_1 & -5x_2 & -10x_3 & = 5 \\ x_1 & +3x_2 & +3x_3 & = -\cancel{10}\ 2 \end{array}$$

$$\left[\begin{array}{ccc|c} -3 & -8 & -13 & 7 \\ -2 & -5 & -10 & 5 \\ 1 & 3 & 3 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ -2 & -5 & -10 & 5 \\ -3 & -8 & -13 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ 0 & 1 & -4 & 1 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$x_3 = t$

$x_2 - 4t = 1$

so $x_2 = 1 + 4t$

$x_1 + 3(1+4t) + 3t = -2$

$x_1 = -2 - 3 - 15t$

$= -5 - 15t$

$x_1 = -5 - 15t$

$x_2 = 1 + 4t$

$x_3 = t$

6. (6pts) Find a matrix B so that $B\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4c & 4d \\ -a & -b \end{bmatrix}$ for every 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We must have $\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

i.e. $\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

So $B = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

Indeed,

$\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4c & 4d \\ -a & -b \end{bmatrix}$

7. (6pts) The matrix B was obtained by applying a row operation to matrix A. Find the elementary matrix E so that $EA = B$.

$A = \begin{bmatrix} 4 & 3 \\ -1 & 5 \\ 7 & 4 \end{bmatrix} \cdot (-2) \quad B = \begin{bmatrix} 4 & 3 \\ -1 & 5 \\ -1 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \leftarrow$ obtained from I by adding $-2\cdot$ first row to third row.

8. (10pts) Below is the augmented matrix of a system of linear equations. Determine the c's for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. Note that no row operations are needed.

$$A = \left[\begin{array}{ccc|c} 1 & -7 & 3 & 5 \\ 0 & c^2-c-2 & c+1 & 5c-10 \\ 0 & 0 & 1 & c+3 \end{array}\right]$$

$c^2-c-2=0$

$(c-2)(c+1)=0$

$c=-1,2$

The system never has infinitely many solutions.

a) When $c \neq -1, 2$, we can divide 2nd row by c^2-c-2, giving us 3 leading 1's, so we get one solution.

c) When $c=-1$ we have

$\left[\begin{array}{ccc|c} 1 & -7 & 3 & 5 \\ 0 & 0 & 0 & -15 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow$ inconsistent, so no solutions

When $c=2$, we have

$-3 \left[\begin{array}{ccc|c} 1 & -7 & 3 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -7 & 3 & 5 \\ 0 & 0 & 0 & -15 \\ 0 & 0 & 1 & 5 \end{array}\right]$

also no solutions

9. (16pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{u} \in \mathbf{R}^3$ is orthogonal to all three standard unit vectors $(1,0,0)$, $(0,1,0)$, $(0,1,0)$, then $\mathbf{u} = \mathbf{0}$.
b) An $n \times n$ matrix with at least one zero entry is not invertible.
c) If A is a 3×3 matrix with at least 1 non-zero entry, then the solution set of the linear system $A\mathbf{x} = \mathbf{0}$ has at most 2 parameters.

a) Let $\vec{u} = (a, b, c)$. Then

True $0 = \vec{u} \cdot (1,0,0) = a$, $0 = \vec{u} \cdot (0,1,0) = b$, $0 = \vec{u} \cdot (0,0,1) = c$

Hence $a = b = c = 0$, so $u = \vec{0}$.

b) False I is invertible yet it has many zeros.

c) If A has at least one nonzero entry, its RRE form will have at least one leading 1. This means there are at most two columns without leading 1's, so at most 2 parameters in the system $A\vec{x} = \vec{0}$.

Bonus. (10pts) Find angle θ so that the matrix A satisfies $A^2 = -I$.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -2\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\cos 2\theta = -1$ so $2\theta = \pi$, $\theta = \frac{\pi}{2}$ Thus, $\theta = \frac{\pi}{2}$

$\sin 2\theta = 0$ so $2\theta = 0, \pi$ $\theta = 0, \pi/2$

Indeed, the square of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is $-I$.