Calculus 1 — Exam 1   **MAT 250, Spring 2012 — D. Ivanšić**

Name: ____________________

Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -1^-} f(x) =$

$\lim_{x\to -1^+} f(x) =$

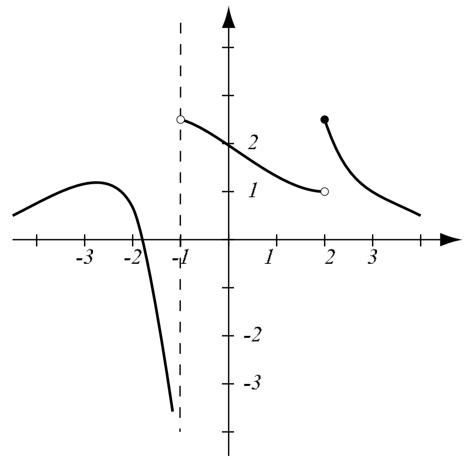
$\lim_{x\to 2-} f(x) =$

$\lim_{x\to 2^+} f(x) =$

$\lim_{x\to 2} f(x) =$

$f(2) =$

List points where f is not continuous and justify why it is not continuous at those points.



2. (4pts) Find the following limit algebraically (no need to justify, other than showing the computation).

$\lim_{x\to 2} \cos(x^2 + 5x - 14) =$

3. (8pts) Let $\lim_{x\to 5} f(x) = -7$ and $\lim_{x\to 5} g(x) = 3$. Use limit laws to find the limit below and show each step.

$$\lim_{x\to 5} \frac{x - 3g(x) - 2}{f(x)\cdot g(x) + x} =$$

4. (14pts) Let $f(x) = \dfrac{3x^2 + x - 1}{x - 3}$.

a) Find the domain of f.

b) Explain, using continuity laws, why the function is continuous on its domain.

c) At points of discontinuity, state the type of discontinuity (jump, infinite, removable) and justify.

5. (16pts) The temperature in degrees Celsius of a rotisserie chicken left to cool is given by $f(t) = 0.15t^2 - 6t + 100$, where t is in minutes.

a) Find the average rates of change of temperature of the chicken over six short intervals of time, three of them beginning with 4, and three ending with 4. Show the table of values. What are the units?

b) Use the information in a) to find the instantaneous rate of cooling of the chicken at $t = 4$. What are the units?

Drool here as needed

6. (14pts) Find an interval of length at most 0.01 that contains the solution of the equation $\ln x = 7 - x$. Use the Intermediate Value Theorem to justify why your interval contains the solution.

7. (16pts) Consider the limit $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$.

a) Can you use a limit law to find the limit? Why or why not?

b) Fill out the table of values below. What do you think is the limit to six decimals accuracy?

c) Explain the unusual values at the bottom of the table.

x	$\frac{\cos x - 1}{x^2}$
0.1	
0.01	
0.001	
10^{-4}	
10^{-5}	
10^{-6}	
10^{-7}	

8. (12pts) Draw the graph of a function, defined on the interval $(-3, 4)$ that exhibits the following features:

$\lim_{x\to -3^+} f(x) = \infty$

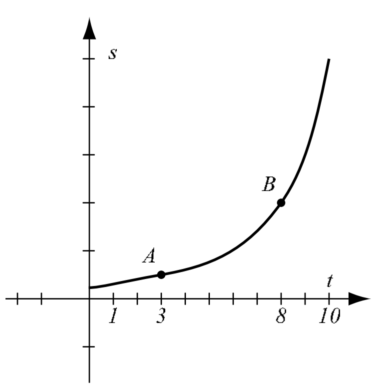
$\lim_{x\to 4^-} f(x) = 3$

$\lim_{x\to 2^-} f(x) = -1$

$\lim_{x\to 2^+} f(x) = 5$

$f(x)$ is right-continuous at $x = 2$

Bonus. (10pts) Below is the graph of the position of a car t minutes after noon. Arrange the numbers a, b, c and d in increasing order (some may be equal) and justify.
a =slope of secant line through points A and B
b =instantaneous velocity at time $t = 3$
c =slope of tangent line at point B.
d =average velocity over the interval $[3, 8]$.



Calculus 1 — Exam 2
MAT 250, Spring 2012 — D. Ivanšić

Name: ____________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left(2x^7 - \dfrac{5}{x^3} + \sqrt[4]{x^7} + e^2\right) =$

2. (6pts) $\dfrac{d}{dt}(t^2 + yt)e^t =$

3. (8pts) $\dfrac{d}{dx}\dfrac{3x-1}{x^3 - 5x^2 + 17} =$

4. (9pts) $\dfrac{d}{dw}\dfrac{w + \sqrt[4]{w}}{w - \sqrt[4]{w}} =$

5. (6pts) Let $h(x) = \dfrac{f(x) + g(x)}{f(x)g(x)}$. Find the general expression for $h'(x)$ and simplify.

Find the following limits algebraically.

6. (5pts) $\lim_{x\to 3} \dfrac{x^2+x-12}{x^2-10x+21} =$

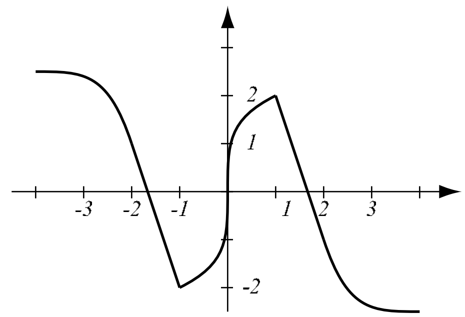
7. (7pts) $\lim_{x\to 25} \dfrac{5-\sqrt{x}}{25-x} =$

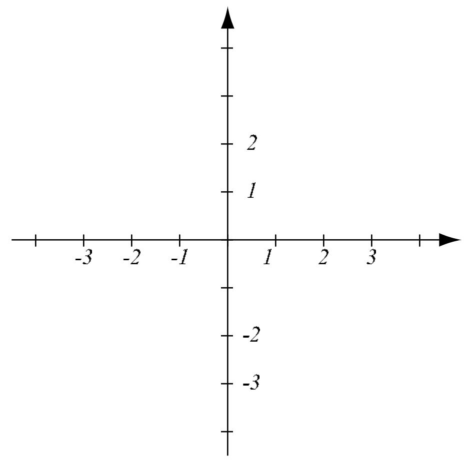
8. (7pts) $\lim_{x\to 0} \dfrac{\sin(3x)}{x^2-x} =$

9. (10pts) Find $\lim_{x\to 0^+} x^3\left(4+\sin^2\left(\dfrac{1}{x}\right)\right)$. Use the theorem that rhymes with what unkind children do to their peers.

10. (12pts) The graph of the function $f(x)$ is shown at right.

a) Find the points where $f'(a)$ does not exist.
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?





11. (16pts) Let $f(x) = \dfrac{x}{x+1}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(1, \frac{1}{2})$.

12. (8pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Use the infomation above and differentiation formulas to find the limit.

$$\lim_{x \to 32} \frac{\sqrt[5]{x} - 2}{x - 32}$$

Bonus. (10pts) We have indicated how to prove $(x^n)' = nx^{n-1}$ for $n \geq 0$. Show that the formula works for integers $n < 0$ as follows: set $n = -k$, and develop the rule for the derivative of x^{-k} with the help of the quotient rule and the rule for positive exponents.

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[0]Total points: 200

Calculus 1 — Exam 3
MAT 250, Spring 2012 — D. Ivanšić

Name: ______________________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{du}\left(e^{2u}\cos u\right) =$

2. (4pts) $\dfrac{d}{dx}\, e^{\tan x} =$

3. (7pts) $\dfrac{d}{d\theta}\left(\sec^2\theta - \tan^2\theta\right) =$

4. (7pts) $\dfrac{d}{dx}\cos\left(\sqrt[3]{x^3 - 3x^2 + 14}\right) =$

5. (8pts) $\dfrac{d}{d\theta}\dfrac{\sin\theta}{\cos^2\theta} =$

6. (8pts) Use implicit differentiation to find y'.

$\sin(x + y) = x\cos y$

7. (10pts) The circle of radius 3 centered at the origin has equation $x^2 + y^2 = 9$. Use implicit differentiation to find the equation of the tangent line at point $(-\sqrt{3}, -\sqrt{6})$. Draw the picture of the circle and the tangent line.

8. (14pts) A lemon is thrown from ground level at initial velocity 20m/s.
a) Write the formula for the position of the lemon at time t (you may assume $g \approx 10$).
b) Write the formula for the velocity of the lemon at time t.
c) What is the lemon's velocity when it is at its highest point?
d) When does the lemon reach its highest point? What is the highest point?
e) What is the acceleration when the lemon is at its highest point? How about at time $t = 0$?

9. (14pts) If you bungee-jump with a cord of length x meters, the time in seconds you will spend free-falling (that is, until the bungee cord engages) is given by $t = \dfrac{\sqrt{x}}{\sqrt{5}}$.

a) Find the free-falling time for a bungee cord of length 45m.
b) Find the ROC of time with respect to cord length when $x = 45$ (units?).
c) Use b) to estimate the change in time if cord length increases by 10m.
d) Use c) to estimate the free-falling time for $x = 55$m and compare to the actual value of 3.3166s.

10. (12pts) Let $f(x) = x^{-4}$.
a) Find the first four derivatives of f.
b) Find the general formula for $f^{(n)}(x)$.

11. (10pts) A remote-controlled bunny's position $s(t)$ is tracked for 10 seconds. Draw the graph of its position function if we know the following:
— $s(0) = 0$, $s(10) = 4$
— it moves forward on interval $(0, 6)$
— it moves backwards on interval $(6, 10)$
— it accelerates on interval $(0, 3)$
— it decelerates on interval $(3, 6)$
— it moves at a steady velocity on interval $(6, 10)$.

Bonus. (10pts) Let $f(x) = x^2 e^x$.
a) Find the first five derivatives of f.
b) Find the pattern for $f^{(n)}(x)$.

Calculus 1 — Exam 4
MAT 250, Spring 2012 — D. Ivanšić

Name: ______________________

Show all your work!

Differentiate and simplify where appropriate:

1. (5pts) $\dfrac{d}{dx}\, 13^{x^2-5x+7} =$

2. (8pts) $\dfrac{d}{dt} \ln \sqrt[5]{\dfrac{t^2}{3t+2}} =$

3. (8pts) $\dfrac{d}{du}\left(u \arctan u - \dfrac{1}{2}\ln(1+u^2)\right) =$

4. (7pts) (note this is not a product, let $x > 0$) $\dfrac{d}{dx} \arcsin\left(\sqrt{1-x^2}\right) =$

5. (8pts) Draw the graph of a function that is continuous and differentiable on $[2, 9]$ which satisfies:

$f'(x) > 0$ on $(2, 4)$
$f'(x) < 0$ on $(4, 7)$
$f'(x) > 0$ on $(7, 9)$
$f(4) = 5,\ f(7) = 3$

6. (12pts) Use Rolle's Theorem to show that the equation $x^3 + e^x = 0$ has at most one solution.

7. (14pts) Let $f(x) = \cos^3 x - \sin^3 x$. Find the absolute minimum and maximum values of f on the interval $[0, 2\pi]$.

8. (10pts) Use logarithmic differentiation to find the derivative of $y = (\cos x)^{\cos x}$.

9. (12pts) Let $f(x) = x^2 - 8x + 15$, $x \leq 4$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find $g'(3)$.

10. (16pts) The angle of elevation is the angle between the ground and the line joining an object with the observer. An outside elevator that is descending at rate 2 meters per second is watched by an observer located on the ground 300 meters from the foot of the building. At what rate is the angle of elevation changing when the elevator is 60 meters above ground?

Bonus. (10pts) Let $f(x)$ be function defined on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ which satisfies: $f'(x) = \cos^2 x$ and $f\left(\frac{\pi}{6}\right) = 0$. Use the Mean Value Theorem to show that $\frac{1}{4}x - \frac{\pi}{24} \leq f(x) \leq \frac{3}{4}x - \frac{\pi}{8}$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

Calculus 1 — Exam 5
MAT 250, Spring 2012 — D. Ivanšić

Name: ____________________

Show all your work!

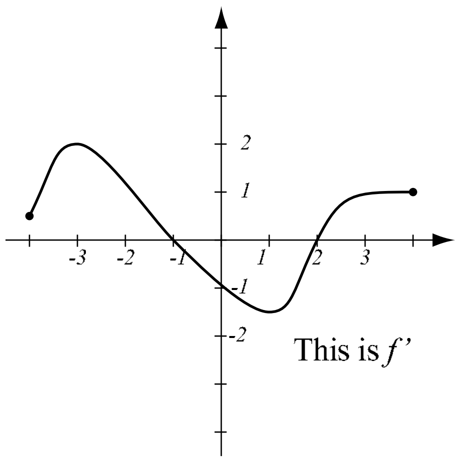
Find the limits. Use L'Hopital's rule where appropriate.

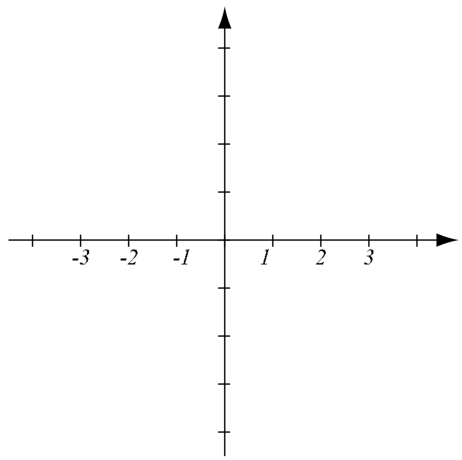
1. (8pts) $\displaystyle\lim_{x\to\infty} \frac{7x^2 - 3x + 4}{\sqrt{3x^4 - 4x^3 + 5}} =$

2. (8pts) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2} =$

3. (10pts) $\displaystyle\lim_{x\to 0+} (\cos x)^{\frac{1}{x}} =$

4. (14pts) Let f be continuous on $[-4, 4]$. The graph of its derivative f' is drawn below. Use the graph to answer:
a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?
b) What are the intervals of concavity of f? Where does f have inflection points?
c) Use the information gathered in a) and b) to draw one possible graph of f at right.





5. (18pts) Let $f(x) = \dfrac{\ln x}{x^3}$, $x > 0$.
a) Find the intervals of concavity and points of inflection for f.
b) Find $\lim_{x\to\infty} f(x)$, and use it, along with concavity, to draw the graph of f for $x > 10$. (You don't need to investigate where f is increasing or decreasing, just draw the right tail-end of f.)

6. (26pts) Let $f(x) = \dfrac{x}{x^2+9}$. Draw an accurate graph of f by following the guidelines.

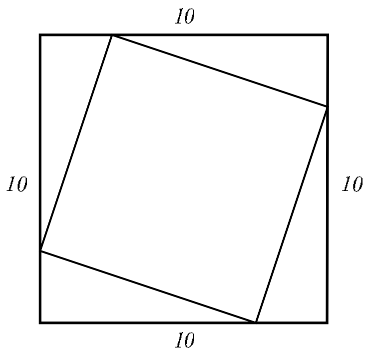
a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

d) Use information from a)–d) to sketch the graph.

7. (16pts) A square is inscribed into a larger square with side length 10, as in the picture.
a) Draw two more possibilities for the inscribed square.
b) Find the inscribed square that has the minimal area.



Bonus. (10pts) Show that $\ln x$ grows slower than any root function. That is, show that for any integer $n > 0$, $\lim_{x\to\infty} \frac{\ln x}{\sqrt[n]{x}} = 0$.

Calculus 1 — Exam 6
MAT 250, Spring 2012 — D. Ivanšić

Name: ____________________

Show all your work!

Find the following antiderivatives.

1. (3pts) $\displaystyle\int e^{3x+2}\,dx =$

2. (7pts) $\displaystyle\int \frac{x^2-4x}{\sqrt{x}}\,dx =$

3. (4pts) $\displaystyle\int \sec^2(3\theta)\,d\theta =$

4. (16pts) Find $\int_0^3 |x-2|\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Fundamental Theorem of Calculus (you will have to break it up into two integrals).

5. (6pts) Evaluate: $\sum_{i=3}^{100}(3i-2) =$

Use the substitution rule in the following integrals:

6. (9pts) $\int x^2(x^3-1)^{\frac{3}{2}}\,dx =$

7. (9pts) $\int_e^{e^2} \frac{1}{x\ln x}\,dx =$

8. (9pts) $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^2\theta}\,d\theta =$

9. (8pts) The velocity of a vibrating spring is $v(t) = 13 \sin 2t$ (in centimeters per second). Find its position function $s(t)$ if $s(0) = 12$ centimeters.

10. (21pts) The function $f(x) = x^2$, $0 \leq x \leq 2$ is given.
a) Write down the expression that is used to compute R_6. Then compute R_6.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does R_6 represent? Does it over- or underestimate the area under the curve.
c) Using the Fundamental Theorem of Calculus, evaluate $\int_0^2 x^2\,dx$. How far off is R_6?

11. (8pts) Show that $\frac{\pi}{12} \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx \leq \frac{\sqrt{3}\pi}{12}$ **without** evaluating the integral.

Bonus. (10pts) The rate at which water flows into a tank is given by the formula $1 - \frac{1}{2}t$ liters per minute. At time $t = 0$, there were 5 liters of water in the tank.
a) When is the tank filling with water, and when is it draining.?
b) How much water got added (or drained) from the tank from $t = 0$ to $t = 6$?
c) How much water is in the tank when $t = 6$?

Calculus 1 — Exam 7
MAT 250, Spring 2012 — D. Ivanšić

Name: ____________________

Show all your work!

1. (10pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

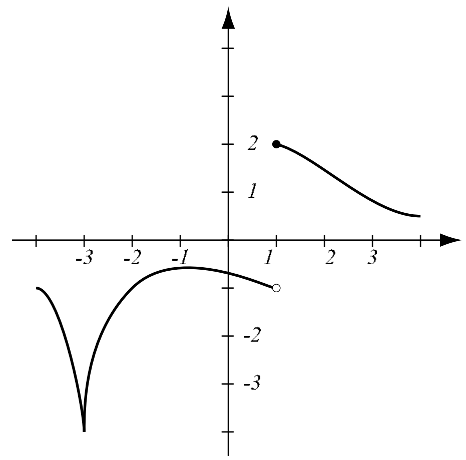
$\lim_{x\to 1^-} f(x) =$

$\lim_{x\to 1^+} f(x) =$

$\lim_{x\to 1} f(x) =$

$f(1) =$

$\lim_{x\to -3} f(x) =$



List points where f is not continuous and explain why.

List points where f is not differentiable and explain why.

2. (12pts) Find the following limit a) algebraically and b) using L'Hospital's rule.

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} =$$

3. (10pts) Let $f(x) = xe^{-x}$. Find the absolute minimum and maximum values of f on the interval $[0,2]$.

4. (12pts) Use implicit differentiation to find the equation of the tangent line to the curve $3x^2 + 4y^2 + 3xy = 25$ at point $(1, 2)$.

5. (12pts) Researchers have found that the volume of a certain marine organism is linked to its surface area via the formula $V = \dfrac{A^{\frac{3}{2}}}{12}$.

a) Find the volume of the organism if its surface area is 9cm^2.

b) Find the ROC of volume with respect to surface area when $A = 9$ (units?).

c) Use b) to estimate the change in volume if surface area increases by 2cm^2.

d) Use c) to estimate the volume of the organism whose surface area is 11cm^2 and compare to the actual value of 3.0402cm^3.

6. (24pts) Let $f(x) = \dfrac{x^2+12}{x-4}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

7. (10pts) Use logarithmic differentiation to find the derivative of $y = \left(1 + \dfrac{1}{x}\right)^x$.

8. (10pts) Let $f(x) = \ln x$.
a) Find the first four derivatives of f.
b) Find the general formula for $f^{(n)}(x)$.

9. (12pts) Among all rectangles of perimeter 20, find the one with the largest area. Show that the area is, indeed, maximal at the point you found.

10. (8pts) Find $f(x)$ if $f'(x) = x^2(x + \sqrt{x})$, if $f(1) = 7$.

11. (8pts) Consider the integral $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\theta \, d\theta$.

a) Use a picture to determine whether this definite integral is positive or negative.
b) Evaluate the integral and verify your conclusion from a).

12. (8pts) Use the substitution rule to find the integral:

$$\int_3^{12} \frac{x-1}{\sqrt[3]{x^2 - 2x + 5}} \, dx =$$

13. (14pts) Luke Skywalker finds himself in a 4 meters wide rectangular garbage compactor containing 30m^3 of water. The side walls are closing in, causing its length to decrease at rate 1 meter per minute and the water level to rise (width stays constant). How fast is the water level increasing when depth is 1.5 meters?

Bonus. (15pts) Let $f(x) = \frac{1}{2}x + \sin x$.

a) On the interval $[0, 4\pi]$, where is the function increasing or decreasing?

b) Show that $\lim_{x\to\infty} f(x) = \infty$. Use the theorem that rhymes with gentle air movement.