Calculus 1 — Exam 7
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

1. (10pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

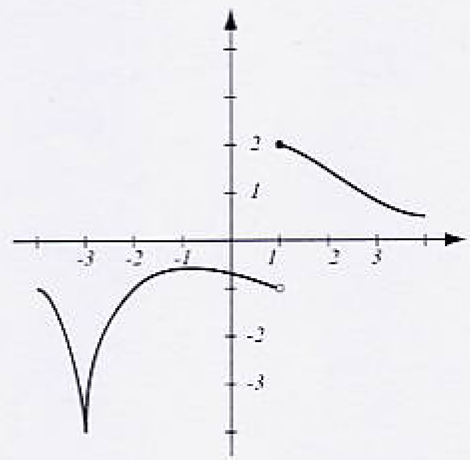
$\lim_{x\to 1^-} f(x) = -1$

$\lim_{x\to 1^+} f(x) = 2$

$\lim_{x\to 1} f(x) =$ d.n.e (one-side limits not equal)

$f(1) = 2$

$\lim_{x\to -3} f(x) = -4$



List points where f is not continuous and explain why.

At $x=1$, $\lim_{x\to 1} f(x)$ d.n.e.

List points where f is not differentiable and explain why.

$x=1$, not even continuous there

$x=-3$, sharp point on graph

2. (12pts) Find the following limit a) algebraically and b) using L'Hospital's rule.

$$\lim_{x\to 3} \frac{\overset{\to 0}{\sqrt{x}-\sqrt{3}}}{\underset{\to 0}{x-3}} = \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3}\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim_{x\to 3}\frac{\cancel{x-3}}{(\cancel{x-3})(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3}\frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$\overset{L'H}{=} \lim_{x\to 3}\frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2\sqrt{3}}$$ (same answer)

3. (10pts) Let $f(x) = xe^{-x}$. Find the absolute minimum and maximum values of f on the interval $[0,2]$.

Critical pts:

$f'(x) = 1\cdot e^{-x} + x\cdot e^{-x}(-1)$

$= e^{-x}(1-x)$ defined everywhere

$\underbrace{e^{-x}}_{\text{always} > 0}(1-x) = 0$

$1-x=0$

$x=1$

x	$f(x)$	
1	$e^{-1}=\frac{1}{e}$	max
0	0	min
2	$2e^{-2}=\frac{2}{e^2}$	

$\frac{2}{e^2} \overset{?}{<} \frac{1}{e}$ $\quad |\cdot e^2$ $\quad 2 \overset{?}{<} e$ yes, so $\frac{2}{e^2} < \frac{1}{e}$

4. (12pts) Use implicit differentiation to find the equation of the tangent line to the curve $3x^2 + 4y^2 + 3xy = 25$ at point $(1, 2)$.

$$3x^2 + 4y^2 + 3xy = 25 \quad \Big| \frac{d}{dx}$$

$$6x + 8yy' + 3(1\cdot y + x\cdot y') = 0$$

$$8yy' + 3xy' = -6x - 3y$$

$$y'(8y + 3x) = -6x - 3y$$

$$y' = \frac{-6x - 3y}{8y + 3x}$$

$$y'\Big|_{(1,2)} = \frac{-6-6}{16+3} = -\frac{12}{19}$$

Eq. of tan. line:

$$y - 2 = -\frac{12}{19}(x - 1)$$

$$\frac{12}{19} + 2 = \frac{50}{19}$$

$$y = -\frac{12}{19}x + \frac{50}{19}$$

5. (12pts) Researchers have found that the volume of a certain marine organism is linked to its surface area via the formula $V = \dfrac{A^{\frac{3}{2}}}{12}$.

a) Find the volume of the organism if its surface area is 9cm^2.

b) Find the ROC of volume with respect to surface area when $A = 9$ (units?).

c) Use b) to estimate the change in volume if surface area increases by 2cm^2.

d) Use c) to estimate the volume of the organism whose surface area is 11cm^2 and compare to the actual value of 3.0402cm^3.

a) $V(9) = \frac{1}{12} 9^{\frac{3}{2}} = \frac{1}{12}\cdot 27 = \frac{27}{12} = \frac{9}{4}$

b) $\frac{dV}{dA} = \frac{1}{12}\cdot\frac{3}{2}A^{\frac{1}{2}} = \frac{1}{8}A^{\frac{1}{2}}$ $\quad \frac{dV}{dA}\Big|_{A=9} = \frac{1}{8}9^{\frac{1}{2}} = \frac{3}{8}$ cm³/cm²

c) $\Delta V \approx V'(9)\cdot\Delta A = \frac{3}{8}\cdot 2 = \frac{6}{8} = \frac{3}{4}$ cm³

d) $V(11) \approx V(9) + \Delta V = \frac{9}{4} + \frac{3}{4} = \frac{12}{4} = 3$, pretty close to 3,0402

6. (24pts) Let $f(x) = \dfrac{x^2+12}{x-4}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

$$y' = \frac{2x(x-4)-(x^2+12)\cdot 1}{(x-4)^2}$$

$$= \frac{2x^2-8x-x^2-12}{(x-4)^2}$$

$$= \frac{x^2-8x-12}{(x-4)^2}$$

$$y'' = \frac{(2x-8)(x-4)^2-(x^2-8x-12)2(x-4)\cdot 1}{(x-4)^4}$$

$$= \frac{\cancel{(x-4)}\big((2x-8)(x-4)-2(x^2-8x-12)\big)}{(x-4)^{\cancel{4}\,3}}$$

$$= \frac{2x^2-16x+32-2x^2+16x+24}{(x-4)^3}$$

$$= \frac{56}{(x-4)^3}$$

a) crit. pts for y':

$x^2-8x-12=0$ $\qquad$ $x-4=0$

$x=4$

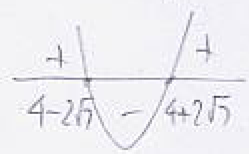
$$x = \frac{8\pm\sqrt{64-4\cdot1\cdot(-12)}}{2}$$

16·7

$$= \frac{8\pm\sqrt{112}}{2} = \frac{8\pm4\sqrt7}{2} = 4\pm2\sqrt7$$

denominator of f' is always positive, so sign only depends on $x^2-8x-12$

$y=x^2-8x-12$



	$-4-2\sqrt7$		4		$4+2\sqrt7$	
f' +	0	−	ND	−	0	+
f ↗	loc max	↘	ND	↘	loc min	↗

b) 2nd order crit. pts,

$x-4=0$, $x=4$

$y=x-4$

		4	
f''	−	ND	+
f	CD	ND	CU

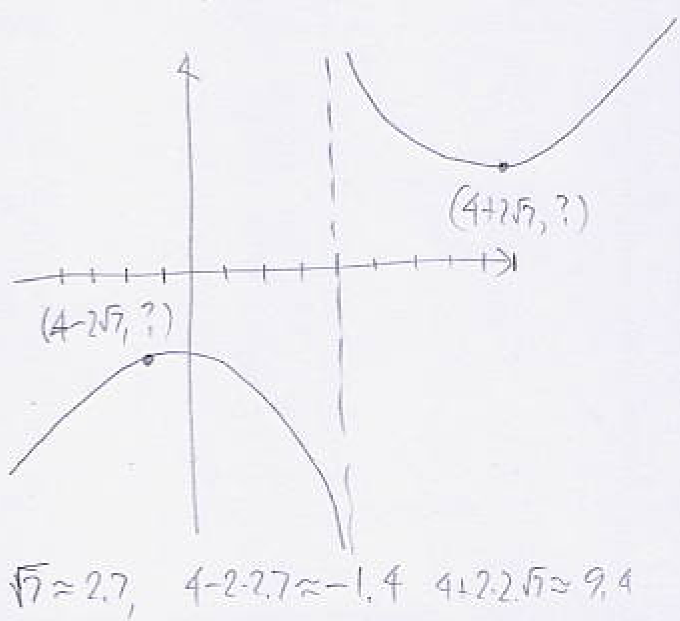
c) $$\lim_{x\to\infty} \frac{x^2-12}{x-4} = \lim_{x\to\infty} \frac{x^2\left(1-\frac{12}{x^2}\right)}{x\left(1-\frac{4}{x}\right)}$$

$\to -\infty$ $\qquad$ $\frac{12}{x^2} \to 0$, $\frac{4}{x} \to 0$

$$= \lim_{x\to\infty} x\cdot 1 = \infty \quad -\infty$$

$\to -\infty$

d) Graph has vertical asymptote at $x=4$



$\sqrt7 \approx 2.7$, $\quad 4-2\cdot2.7 \approx -1.4 \quad 4+2\cdot2\sqrt7 \approx 9.4$

7. (10pts) Use logarithmic differentiation to find the derivative of $y = \left(1 + \frac{1}{x}\right)^x$.

$\ln y = \ln\left(1+\frac{1}{x}\right)^x$

$\ln y = x \ln\left(1+\frac{1}{x}\right) \quad \Big| \frac{d}{dx}$

$\frac{y'}{y} = 1 \cdot \ln\left(1+\frac{1}{x}\right) + \cancel{x} \frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{\cancel{x^2}}$

$= \ln\left(1+\frac{1}{x}\right) - \frac{1}{1+\frac{1}{x}} \cdot \frac{1}{x}$

$\frac{y'}{y} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$y' = \left(1+\frac{1}{x}\right)^x \left(\ln(1+x) - \frac{1}{x+1}\right)$

8. (10pts) Let $f(x) = \ln x$.
a) Find the first four derivatives of f.
b) Find the general formula for $f^{(n)}(x)$.

a) $y = \ln x$

$y' = x^{-1}$

$y'' = (-1)x^{-2}$

$y''' = (-1)(-2)x^{-3}$

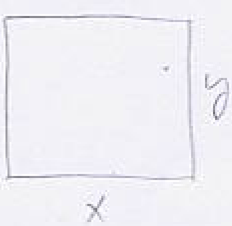
$y^{(4)} = (-1)(-2)(-3)x^{-4}$

b) $y^{(n)} = (-1)(-2) \ldots (-(n-1))x^{-n}$

$= (-1)^{n-1} 1 \cdot 2 \cdots (n-1) x^{-n}$

$y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$

9. (12pts) Among all rectangles of perimeter 20, find the one with the largest area. Show that the area is, indeed, maximal at the point you found.



$2x + 2y = 20$

$x + y = 10$

$y = 10 - x$

$A = xy = x(10-x) = -x^2 + 10x$

Job: maximize $A(x) = -x^2 + 10x$

on $[0, 10]$

$2x \le 20$

so $x \le 10$

$A'(x) = -2x + 10$

$-2x + 10 = 0$

$x = 5$

x	$A(x)$
5	25
0	0
10	0

maximal

Greatest area occurs when rectangle is a square.

10. (8pts) Find $f(x)$ if $f'(x) = x^2(x+\sqrt{x})$, if $f(1) = 7$.

$f'(x) = x^3 + x^{\frac{5}{2}}$

$f(x) = \frac{x^4}{4} + \frac{2}{7}x^{\frac{7}{2}} + C$

$7 = f(1) = \frac{1}{4} + \frac{2}{7} + C$

$7 = \frac{15}{28} + C$

$C = 7 - \frac{15}{28} = \frac{196}{28} - \frac{15}{28} = \frac{181}{28}$

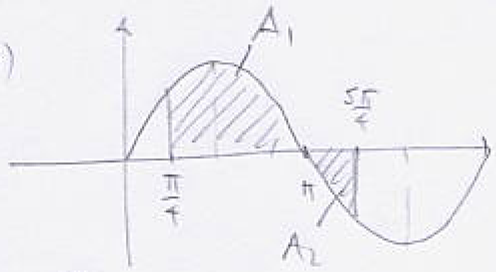
$f(x) = \frac{x^4}{4} + \frac{2}{7}x^{\frac{7}{2}} + \frac{181}{28}$

11. (8pts) Consider the integral $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\theta \, d\theta$.

a) Use a picture to determine whether this definite integral is positive or negative.

b) Evaluate the integral and verify your conclusion from a).

a)



$\int_{\pi/4}^{\frac{5\pi}{4}} \sin\theta \, d\theta = A_1 - A_2 > 0$ since $A_1 > A_2$

b) $\int_{\pi/4}^{5\pi/4} \sin\theta \, d\theta$

$= -\cos\theta \Big|_{\pi/4}^{5\pi/4} = -\left(\cos\frac{5\pi}{4} - \cos\frac{\pi}{4}\right)$

$= -\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2} > 0,$

12. (8pts) Use the substitution rule to find the integral:

$\int_3^{12} \frac{x-1}{\sqrt[3]{x^2-2x+5}} \, dx = \begin{bmatrix} u = x^2 - 2x + 5 & x = 12,\ u = 144 - 24 + 5 = 125 \\ du = (2x-2)dx & x = 3,\ u = 9 - 6 + 5 = 8 \end{bmatrix} = \int_8^{125} \frac{1}{\sqrt[3]{u}} \frac{1}{2} du$

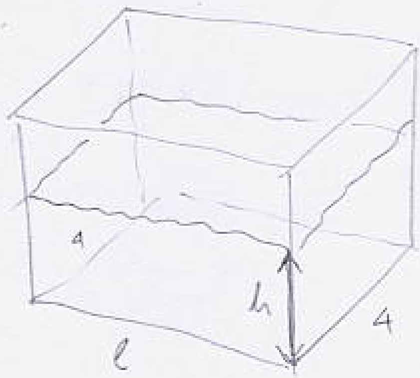
$\frac{1}{2}du = (x-1)\,dx$

$u^{-\frac{1}{3}}$

$= \frac{1}{2}\frac{3}{2}u^{\frac{2}{3}} \Big|_8^{125} = \frac{3}{4}\left(125^{\frac{2}{3}} - 8^{\frac{2}{3}}\right) = \frac{3}{4}(25-4) = \frac{63}{4}$

$(\sqrt[3]{125})^2 - (\sqrt[3]{8})^2$

13. (14pts) Luke Skywalker finds himself in a 4 meters wide rectangular garbage compactor containing 30m^3 of water. The side walls are closing in, causing its length to decrease at rate 1 meter per minute and the water level to rise (width stays constant). How fast is the water level increasing when depth is 1.5 meters?



Need: $\frac{dh}{dt}$ when $h = 1.5$

Know: $\frac{dl}{dt} = -1$ m/min

$4h \cdot l = 30$

$h = \frac{30}{4l} = \frac{15}{2l} \quad \Big| \frac{d}{dt}$

$h' = \frac{15}{2}\left(-\frac{1}{l^2}\right) \cdot l' = -\frac{15 l'}{2l^2}$

When $h = 1.5$

$4 \cdot 1.5 \cdot l = 30$

$6l = 30$

$l = 5$

$h' = -\frac{\overset{3}{\cancel{15}} \cdot (-1)}{2 \cdot \underset{5}{\cancel{25}}} = \frac{3}{2}$ m/min

Bonus. (15pts) Let $f(x) = \frac{1}{2}x + \sin x$.

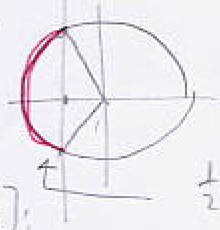
a) On the interval $[0, 4\pi]$, where is the function increasing or decreasing?

b) Show that $\lim_{x\to\infty} f(x) = \infty$. Use the theorem that rhymes with gentle air movement.

a) $f'(x) = \frac{1}{2} + \cos x$

$\frac{1}{2} + \cos x = 0$

$\cos x = -\frac{1}{2}$



$(\cos x < -\frac{1}{2})$

$\frac{1}{2} + \cos x < 0$

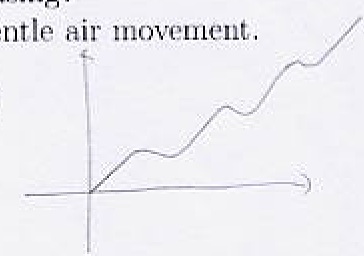
only in this interval

Sol. on $[0, 4\pi]$:

$x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

	0		$\frac{2\pi}{3}$		$\frac{4\pi}{3}$		2π		$\frac{8\pi}{3}$		$\frac{10\pi}{3}$		4π
f'		+	0	−	0	+		+	0	−	0	+	
f		↗	loc. max	↘	loc. min	↗		↗	loc. max	↘	loc. min	↗	

Graph of f is



b) $-1 \le \sin x \le 1$

$\frac{1}{2}x - 1 \le \frac{1}{2}x + \sin x \le \frac{1}{2}x + 1$

$\lim_{x\to\infty} \frac{1}{2}x - 1 = \infty$

$\lim_{x\to\infty} \frac{1}{2}x + 1 = \infty$

since limits are equal, by the squeeze theorem,

$\lim_{x\to\infty} \frac{1}{2}x + \sin x = \infty$