Calculus 1 — Exam 6
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

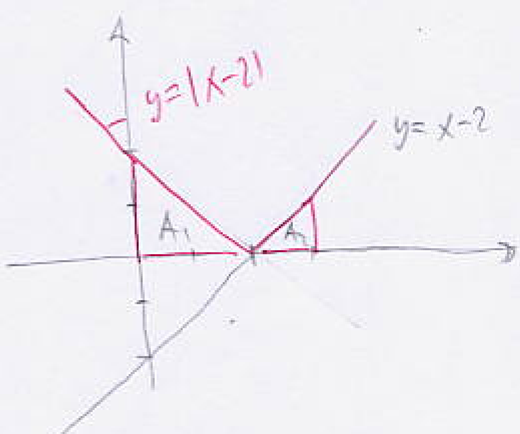
Find the following antiderivatives.

1. (3pts) $\int e^{3x+2}\,dx = \frac{1}{3}e^{3x+2} + C$

2. (7pts) $\int \frac{x^2-4x}{\sqrt{x}}\,dx = \int \frac{x^2-4x}{x^{1/2}}\,dx = \int x^{\frac{3}{2}} - 4x^{\frac{1}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} - 4\cdot\frac{2}{3}x^{\frac{3}{2}} = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + C$

3. (4pts) $\int \sec^2(3\theta)\,d\theta = \frac{1}{3}\tan(3\theta) + C$

4. (16pts) Find $\int_0^3 |x-2|\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Fundamental Theorem of Calculus (you will have to break it up into two integrals).



a) $\int_0^3 |x-2|\,dx = A_1 + A_2 = \frac{1}{2}2\cdot 2 + \frac{1}{2}1\cdot 1$
$= 2 + \frac{1}{2} = \frac{5}{2}$

b) $\int_0^3 |x-2|\,dx = \int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx$

$= \int_0^2 -(x-2)\,dx + \int_2^3 x-2\,dx$

$= -\left(\frac{x^2}{2}-2x\right)\Big|_0^2 + \left(\frac{x^2}{2}-2x\right)\Big|_2^3$

$= -(2-4) - 0 + \frac{1}{2}(3^2-2^2) - 2(3-2)$

$= 2 + \frac{5}{2} - 2 = \frac{5}{2}$ Same as in a) expression when

5. (6pts) Evaluate: $\sum_{i=3}^{100}(3i-2) =$ $\sum_{i=1}^{100}(3i-2) - (\underset{i=1}{1} + \underset{i=2}{4}) = 3\sum_{i=1}^{100} i - \sum_{i=1}^{100} 2 - 5$

$= 3 \cdot \frac{100 \cdot 101}{2} - 2 \cdot 100 - 5 = 150 \cdot 101 - 205 = 15150 - 205 = 14,945$

15000
150

Use the substitution rule in the following integrals:

6. (9pts) $\int x^2(x^3-1)^{\frac{3}{2}}\,dx =$ $\begin{bmatrix} u = x^3-1 \\ du = 3x^2\,dx \\ \frac{1}{3}du = x^2\,dx \end{bmatrix} = \int \frac{1}{3} u^{\frac{3}{2}}\,du = \frac{1}{3}\cdot\frac{2}{5}u^{\frac{5}{2}}$

$= \frac{2}{15}(x^3-1)^{\frac{5}{2}} + C$

7. (9pts) $\int_e^{e^2} \frac{1}{x\ln x}\,dx =$ $\begin{bmatrix} u = \ln x & x=e^2,\ u=2 \\ du = \frac{1}{x}dx & x=e \quad u=1 \end{bmatrix} = \int_1^2 \frac{1}{u}\,du$

$= \ln|u| \Big|_1^2 = \ln 2 - \underbrace{\ln 1}_{=0} = \ln 2$

8. (9pts) $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^2\theta}\,d\theta =$ $\begin{bmatrix} u = \cos\theta & \theta = \frac{\pi}{4},\ u = \frac{\sqrt{2}}{2} \\ du = -\sin\theta\,d\theta & \theta = 0,\ u = 1 \\ -du = \sin\theta\,d\theta & \end{bmatrix} = \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u^2}\,du$

$-u^{-2}$

$= -\frac{u^{-1}}{-1}\Big|_1^{\frac{\sqrt{2}}{2}} = \frac{1}{u}\Big|_1^{\frac{\sqrt{2}}{2}} = \frac{1}{\frac{\sqrt{2}}{2}} - \frac{1}{1} = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

9. (8pts) The velocity of a vibrating spring is $v(t) = 13\sin 2t$ (in centimeters per second). Find its position function $s(t)$ if $s(0) = 12$ centimeters.

$$v(t) = 13\sin(2t)$$

$$s(t) = 13\left(-\frac{\cos(2t)}{2}\right) + C$$

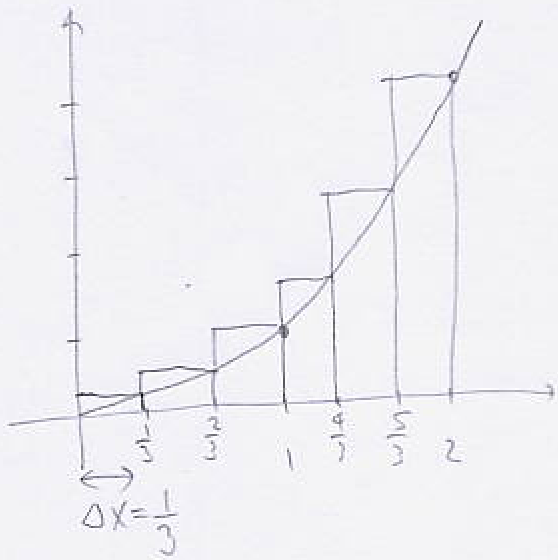
$$12 = s(0) = -\frac{13}{2}\cdot 1 + C$$

$$C = 12 + \frac{13}{2} = \frac{37}{2}$$

$$s(t) = -\frac{13}{2}\cos(2t) + \frac{37}{2}$$

10. (21pts) The function $f(x) = x^2$, $0 \le x \le 2$ is given.
a) Write down the expression that is used to compute R_6. Then compute R_6.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does R_6 represent? Does it over- or underestimate the area under the curve.
c) Using the Fundamental Theorem of Calculus, evaluate $\int_0^2 x^2\,dx$. How far off is R_6?

b)



$$R_6 = \frac{1}{3}\left(\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + 2^2\right)$$

$$= \frac{1}{3}\left(\frac{1+4+16+25}{9} + 5\right)$$

$$= \frac{1}{3}\left(\frac{46}{9} + 5\right) = \frac{1}{3}\,\frac{91}{9} = \frac{91}{27}$$

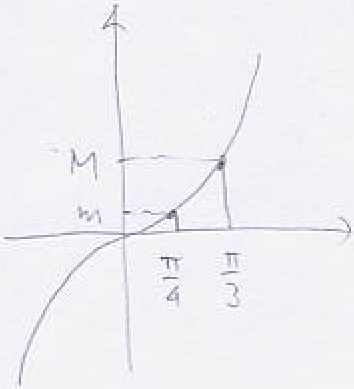
R_6 overestimates the area

R_6 is sum of areas of rectangles.

c) $$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$

$$\frac{8}{3} - \frac{91}{27} = \frac{72-91}{27} = \frac{19}{27} \approx \frac{2}{3},$$ fairly far.

11. (8pts) Show that $\frac{\pi}{12} \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx \le \frac{\sqrt{3}\pi}{12}$ **without** evaluating the integral.



On $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$

$$\tan\frac{\pi}{4} \le \tan x \le \tan\frac{\pi}{3}$$

$$1 \le \tan x \le \sqrt{3}$$

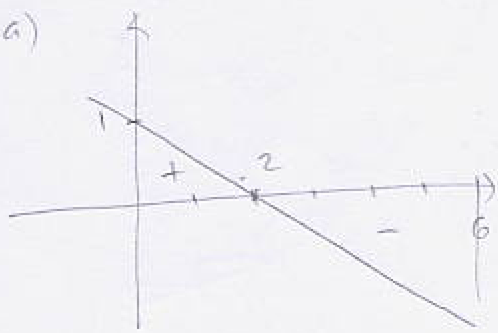
$$1\cdot\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x \, dx \le \sqrt{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x \, dx \le \sqrt{3}\cdot\frac{\pi}{12} \quad \text{as needed.}$$

Bonus. (10pts) The rate at which water flows into a tank is given by the formula $1 - \frac{1}{2}t$ liters per minute. At time $t = 0$, there were 5 liters of water in the tank.
a) When is the tank filling with water, and when is it draining.?
b) How much water got added (or drained) from the tank from $t = 0$ to $t = 6$?
c) How much water is in the tank when $t = 6$?

Let $V(t)$ = volume of water in the tank. $V'(t) = 1 - \frac{1}{2}t$

a)



$1 - \frac{1}{2}t = 0$

$t = 2$

$V'(t) > 0$ on $[0, 2]$, tank filling
$V'(t) < 0$ on $[2, 6]$ tank draining

b) $\Delta V = \int_0^6 V'(t) = \int_0^6 1 - \frac{1}{2}t \, dt$

$= \left(t - \frac{t^2}{4}\right)\Big|_0^6 = 6 - \frac{36}{4} - 0 = 6 - 9 = -3$

3 liters of water drained from $t = 0$ to $t = 6$.

c) $V(6) = V(0) + \Delta V$
$= 5 + (-3) = 2$ liters remain at $t = 6$.