Calculus 1 — Exam 5
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

Find the limits. Use L'Hopital's rule where appropriate.

1. (8pts) $\displaystyle\lim_{x\to\infty} \frac{7x^2-3x+4}{\sqrt{3x^4-4x^3+5}} = \lim_{x\to\infty} \frac{x^2\left(7-\frac{3}{x}+\frac{4}{x^2}\right)}{\sqrt{x^4\left(3-\frac{4}{x}+\frac{5}{x^4}\right)}} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(7-\frac{3}{x}+\frac{4}{x^2}\right)}{\cancel{x^2}\sqrt{3-\frac{4}{x}+\frac{5}{x^4}}}$

$$= \frac{7-0+0}{\sqrt{3-0+0}} = \frac{7}{\sqrt{3}}$$

2. (8pts) $\displaystyle\lim_{x\to0} \frac{\overset{\to 1-1=0}{\cos x - 1}}{\underset{\to 0}{x^2}} \overset{L'H}{=} \lim_{x\to0} \frac{\overset{\to 0}{-\sin x}}{\underset{\to 0}{2x}} \overset{L'H}{=} \lim_{x\to0} \frac{-\cos x}{2} = -\frac{1}{2}$

3. (10pts) $\displaystyle\lim_{x\to0+} (\cos x)^{\frac{1}{x}} = e^0 = 1$

$y = (\cos x)^{\frac{1}{x}}$

$\ln y = \ln(\cos x)^{\frac{1}{x}} = \frac{1}{x}\ln\cos x$

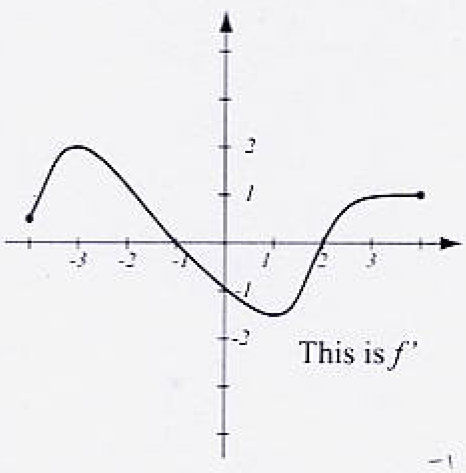
$\displaystyle\lim_{x\to0+} \frac{\ln\cos x}{x} \overset{L'H}{=} \lim_{x\to0+} \frac{\frac{1}{\cos x}(-\sin x)}{1} = \lim_{x\to0+} -\tan x = 0$

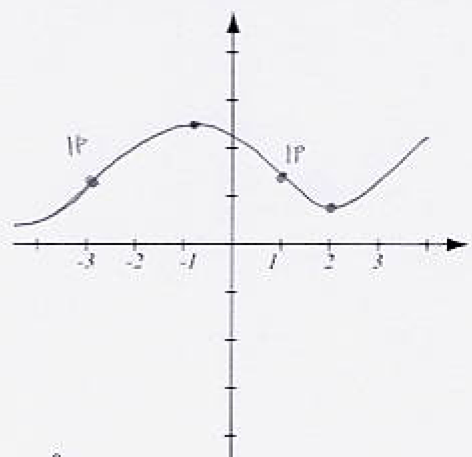
4. (14pts) Let f be continuous on $[-4,4]$. The graph of its derivative f' is drawn below. Use the graph to answer:

a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?

b) What are the intervals of concavity of f? Where does f have inflection points?

c) Use the information gathered in a) and b) to draw one possible graph of f at right.





f incr / decr $\iff f' > 0$ / < 0

f CU / CD $\iff f'$ incr / decr

a) f incr. on $(-4,-1)\cup(2,4)$

f decr. on $(-1,-2)$

local max at $x=-1$

local min at $x=2$

b) f conc up on $(-4,-3)\cup(1,4)$

f conc. dow on $(-3,-1)$

Infl. pt. at $x=-3, 2$

		-1		2	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	loc max	$\searrow$	loc min	$\nearrow$

		-3		1	
f''	$+$		$-$		$+$
f	CU	IP	CD	IP	CU

5. (18pts) Let $f(x) = \dfrac{\ln x}{x^3}$, $x > 0$.

a) Find the intervals of concavity and points of inflection for f.

b) Find $\lim\limits_{x\to\infty} f(x)$, and use it, along with concavity, to draw the graph of f for $x > 10$. (You don't need to investigate where f is increasing or decreasing, just draw the right tail-end of f.)

c) $$f'(x) = \frac{\frac{1}{x}\cdot x^3 - \ln x\cdot 3x^2}{(x^3)^2} = \frac{x^2 - \ln x\cdot 3x^2}{x^6} = \frac{\cancel{x^2}(1-3\ln x)}{x^{\cancel{6}\,4}} = \frac{1-3\ln x}{x^4}$$

$$f''(x) = \frac{\left(-3\cdot\frac{1}{x}\right)x^4 - (1-3\ln x)\cdot 4x^3}{(x^4)^2} = \frac{-3x^3 - (4-12\ln x)x^3}{x^8} = \frac{\cancel{x^3}(-7+12\ln x)}{x^{\cancel{8}\,5}} =$$

$$= \frac{12\ln x - 7}{x^5}$$

	0		$e^{\frac{7}{12}}$	
$12\ln x - 7$		$-$	0	$+$
f		CD		CU

x	$12\ln x - 7$
1	-7
e^1	5

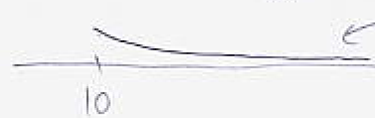
$12\ln x - 7 = 0$ $\qquad x^5 = 0$

$\ln x = \dfrac{7}{12}$ $\qquad x = 0$ (not in domain)

$x = e^{\frac{7}{12}}$

b) $$\lim_{x\to\infty} \frac{\ln x}{x^3} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x\to\infty} \frac{1}{3x^3} = 0$$

Graph:



concave up, horiz. asymptote $y=0$

6. (26pts) Let $f(x) = \dfrac{x}{x^2+9}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–d) to sketch the graph.

$$y' = \frac{1\cdot(x^2+9) - x\cdot 2x}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2} \quad \} > 0$$

$$y'' = \frac{-2x(x^2+9)^2 - (9-x^2)2(x^2+9)\cdot 2x}{(x^2+9)^4}$$

$$= \frac{-\cancel{(x^2+9)}2x\left(x^2+9+2(9-x^2)\right)}{(x^2+9)^{\cancel{4}\,3}}$$

$$= -\frac{2x(27-x^2)}{(x^2+9)^3} = \frac{2x(x^2-27)}{(x^2+9)^3} \quad \} > 0$$

a) $y' = 0$ or not def:

$9 - x^2 = 0$

$x^2 = 9$

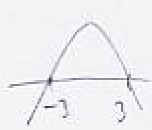
$x = \pm 3$

$(x^2+9)^2 = 0$

$x^2 = -9$

no solution

Sign of y' only depends on $9-x^2$;



		-3		3	
f'	$-$	0	$+$	0	$-$
f	$\searrow$	loc min	$\nearrow$	loc max	$\searrow$

b) $y'' = 0$ or not def (never)

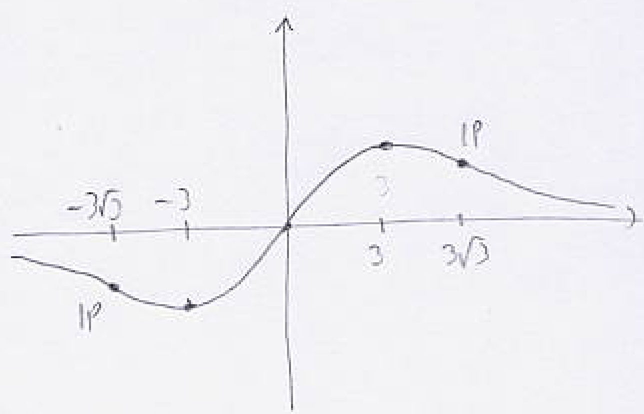
$x(x^2-27) = 0$

$x = 0$ or $x^2 = 27$

$x = \pm\sqrt{27} = \pm 3\sqrt{3}$

Sign of y'' depends on x, x^2-27

		$-3\sqrt{3}$		0		$3\sqrt{3}$	
x	$-$		$-$	0	$+$		$+$
x^2-27	$+$	0	$-$		$-$	0	$+$
f''	$-$		$+$		$-$		$+$
f	CD	IP	CU	IP	CD	IP	CU



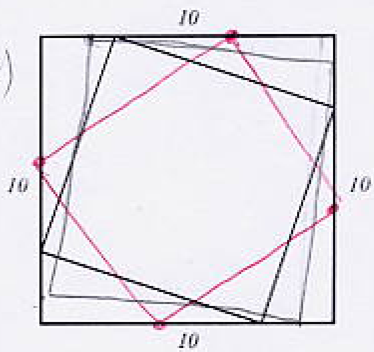
c) $\lim\limits_{x\to\infty} \dfrac{x}{x^2+9} = \lim\limits_{x\to\infty} \dfrac{x}{x^2(1+\frac{9}{x^2})} = \lim\limits_{x\to\infty} \dfrac{1}{x(1+\frac{9}{x^2})} = \dfrac{1}{\infty(1+0)} = \dfrac{1}{\infty} = 0$

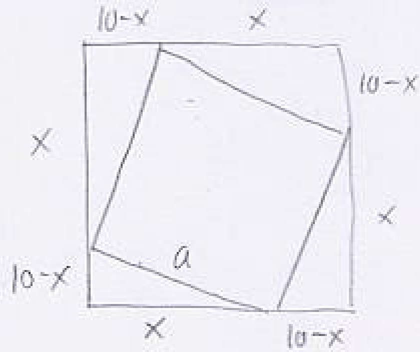
same calculation for $x \to -\infty$.

x	$\frac{x}{x^2+9}$
$-3\sqrt{3}$	$-3\sqrt{3}/36 = -\sqrt{3}/12$
-3	$-3/18 = -1/6$
0	0
3	$\sqrt{3}/12$
$3\sqrt{3}$	$1/6$

7. (16pts) A square is inscribed into a larger square with side length 10, as in the picture.
a) Draw two more possibilities for the inscribed square.
b) Find the inscribed square that has the minimal area.

a)





Area $= a^2 = (10-x)^2 + x^2$

$A(x) = x^2 + (10-x)^2$

Job: minimize $A(x) = x^2 + (10-x)^2$ on $[0, 10]$

$A'(x) = 2x + 2(10-x)(-1)$

$= 2x + 2x - 20$

$= 4x - 20$

$4x - 20 = 0$

$x = 5$

x	$x^2 + (10-x)^2$
0	100
10	100
5	$25+25=50$

Minimal area occurs when $x = 5$.

Bonus. (10pts) Show that $\ln x$ grows slower than any root function. That is, show that for any integer $n > 0$, $\lim_{x\to\infty} \frac{\ln x}{\sqrt[n]{x}} = 0$.

$$\lim_{x\to\infty} \frac{\overset{\to\infty}{\ln x}}{\underset{\to\infty}{x^{1/n}}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{\frac{1}{n}-1}} = \lim_{x\to\infty} \overset{=x^{-1}}{\frac{1}{x}} \cdot n \cdot x^{1-\frac{1}{n}} = \lim_{x\to\infty} n x^{-1+1-\frac{1}{n}} = \lim_{x\to\infty} n x^{-\frac{1}{n}}$$

$$= \lim_{x\to\infty} \frac{n}{x^{1/n}} = \frac{n}{\infty} = 0 \quad \text{for any } n > 0.$$