Calculus 1 — Exam 4
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (5pts) $\dfrac{d}{dx}13^{x^2-5x+7} =$ $\ln 13 \cdot 13^{x^2-5x+7} \cdot (2x-5)$

2. (8pts) $\dfrac{d}{dt}\ln\sqrt[5]{\dfrac{t^2}{3t+2}} =$ $\dfrac{d}{dt}\dfrac{1}{5}\ln\dfrac{t^2}{3t+2} = \dfrac{d}{dt}\dfrac{1}{5}\left(\ln t^2 - \ln(3t+2)\right)$

$= \dfrac{1}{5}\dfrac{d}{dt}\left(2\ln t - \ln(3t+2)\right) = \dfrac{1}{5}\left(\dfrac{2}{t} - \dfrac{3}{3t+2}\right) = \dfrac{2(3t+2)-3t}{5t(3t+2)} = \dfrac{3t+4}{5t(3t+2)}$

3. (8pts) $\dfrac{d}{du}\left(u\arctan u - \dfrac{1}{2}\ln(1+u^2)\right) =$ $\underbrace{1\cdot\arctan u + u\cdot\dfrac{1}{1+u^2} - \dfrac{1}{\cancel{2}}\dfrac{1}{1+u^2}\cdot\cancel{2}u}_{\text{these cancel}}$

$= \arctan u$

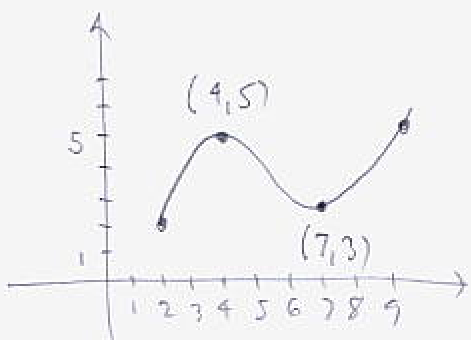
4. (7pts) (note this is not a product, let $x > 0$) $\dfrac{d}{dx}\arcsin\left(\sqrt{1-x^2}\right) =$ $\dfrac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\dfrac{1}{2\sqrt{1-x^2}}\cdot(-2x)$

$= \dfrac{1}{\sqrt{1-(1-x^2)}}\cdot\dfrac{-x}{\sqrt{1-x^2}} = \dfrac{-x}{\underbrace{\sqrt{x^2}}_{=x \text{ since } x>0}\cdot\sqrt{1-x^2}} = -\dfrac{1}{\sqrt{1-x^2}}$

5. (8pts) Draw the graph of a function that is continuous and differentiable on $[2, 9]$ which satisfies:

$f'(x) > 0$ on $(2, 4)$
$f'(x) < 0$ on $(4, 7)$
$f'(x) > 0$ on $(7, 9)$
$f(4) = 5$, $f(7) = 3$

	2		4		7		9
f'		+	0	−	0	+	
f		↗	loc. max	↘	loc. min	↗	



6. (12pts) Use Rolle's Theorem to show that the equation $x^3 + e^x = 0$ has at most one solution.

Suppose $x^3+e^x=0$ has two or more solutions: $a, b, \ldots$

Let $f(x) = x^3+e^x$. Then $f(a)=f(b)=0$. Since x^3+e^x is continuous and differentiable on $\mathbb{R}$, and hence on $[a,b]$, Rolle's theorem applies, guaranteeing a $c \in (a,b)$ s.t. $f'(c)=0$.

However $f'(x) = 3x^2+e^x$. $3x^2+e^x=0$ does not have a solution, since the left side is positive, so we have arrived at a contradiction. This means the original statement is true.

7. (14pts) Let $f(x) = \cos^3 x - \sin^3 x$. Find the absolute minimum and maximum values of f on the interval $[0, 2\pi]$.

Crit. pts:

$$f'(x) = 3\cos^2 x(-\sin x) - 3\sin^2 x\cos x$$

$$= -3\sin x\cos x(\cos x + \sin x)$$

$$-3\sin x\cos x(\cos x+\sin x) = 0$$

$\sin x = 0$ $\quad$ $\cos x = 0$ $\quad$ or $\quad$ $\cos x + \sin x = 0$

$x = 0, \pi, 2\pi$ $\quad$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\cos x = -\sin x$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

When $x=-y$

	x	$\cos^3 x - \sin^3 x$	
endpts	0	$1-0=1$	max
	2π	$1-0=1$	max
	π	$-1-0=-1$	min
	$\frac{\pi}{2}$	$0-1=-1$	min
	$\frac{3\pi}{2}$	$0-(-1)=1$	max
	$\frac{3\pi}{4}$	$\left(-\frac{\sqrt2}{2}\right)^3-\left(\frac{\sqrt2}{2}\right)^3 = \frac{-2\sqrt2-2\sqrt2}{8} = -\frac{\sqrt2}{2}$	
	$\frac{7\pi}{4}$	$\left(\frac{\sqrt2}{2}\right)^3-\left(-\frac{\sqrt2}{2}\right)^3 = \frac{2\sqrt2+2\sqrt2}{8} = \frac{\sqrt2}{2}$	

$\left(\frac{\sqrt2}{2} < 1\right)$

8. (10pts) Use logarithmic differentiation to find the derivative of $y = (\cos x)^{\cos x}$.

$$y = (\cos x)^{\cos x}$$

$$\ln y = \ln (\cos x)^{\cos x}$$

$$\ln y = \cos x \ln(\cos x) \quad \Big| \frac{d}{dx}$$

$$\frac{1}{y} \cdot y' = -\sin x \ln(\cos x) + \cos x \frac{1}{\cos x}(-\sin x) \quad \Big| \cdot y$$

$$y' = y\left(-\sin x \ln(\cos x) - \sin x\right)$$

$$= -\sin x (\cos x)^{\cos x}\left(\ln(\cos x) + 1\right)$$

9. (12pts) Let $f(x) = x^2 - 8x + 15$, $x \leq 4$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find $g'(3)$.

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(2)} = \frac{1}{2x-8}\Big|_{x=2} = \frac{1}{4-8} = -\frac{1}{4}$$

$$x \underset{g}{\overset{f}{\rightleftarrows}} 3$$

Need x s.t. $f(x) = 3$

$$x^2 - 8x + 15 = 3$$

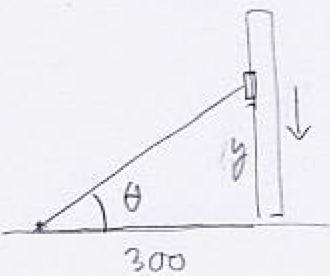
$$x^2 - 8x + 12 = 0$$

$$(x-6)(x-2) = 0$$

$$x = 2, 6$$

not ≤ 4

10. (16pts) The angle of elevation is the angle between the ground and the line joining an object with the observer. An outside elevator that is descending at rate 2 meters per second is watched by an observer located on the ground 300 meters from the foot of the building. At what rate is the angle of elevation changing when the elevator is 60 meters above ground?



Need: θ' when $y = 60$

Know: $y' = -2$ m/s

$$\frac{y}{300} = \tan\theta$$

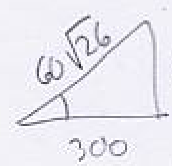
$$y = 300\tan\theta \quad \Big|\ \frac{d}{dt}$$

$$y' = 300\sec^2\theta \cdot \theta'$$

$$\theta' = \frac{y'}{300\sec^2\theta} = \frac{\cos^2\theta\, y'}{300}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

When $y = 60$ we have



$$\sqrt{60^2 + 300^2} = \sqrt{60^2 + (60\cdot 5)^2} = \sqrt{60^2(1+5^2)} = 60\cdot\sqrt{26}$$

$$\cos^2\theta = \frac{300^2}{60^2\cdot 26}$$

$$\theta' = \frac{\frac{300^2}{60^2\cdot 26}\cdot(-2)}{300} = \frac{300^{\cancel{2}}\,(-\cancel{2})}{\cancel{300}\cdot 60^2\cdot \cancel{26}_{13}}$$

$$= -\frac{\cancel{300}^{\,81}}{\cancel{60}\cdot\cancel{60}_{12}\cdot 13} = -\frac{1}{12\cdot 13} = -\frac{1}{156}\ \text{rad/s}$$

Bonus. (10pts) Let $f(x)$ be function defined on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ which satisfies: $f'(x) = \cos^2 x$ and $f\left(\frac{\pi}{6}\right) = 0$. Use the Mean Value Theorem to show that $\frac{1}{4}x - \frac{\pi}{24} \le f(x) \le \frac{3}{4}x - \frac{\pi}{8}$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.

Since $f'(x) = \cos^2 x$, f is differentiable and hence continuous, so MVT applies.

Apply it to interval $\left[\frac{\pi}{6}, x\right]$: then exists a $c \in \left[\frac{\pi}{6}, x\right]$ s.t.

$$f'(c) = \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}}$$

Since $f'(c) = \cos^2 c$, and $c \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ we have

$$\cos^2\frac{\pi}{6} \ge f'(c) \ge \cos^2\frac{\pi}{3}$$

$$\left(\frac{\sqrt{3}}{2}\right)^2 \ge f'(c) \ge \left(\frac{1}{2}\right)^2$$

Then the same is true for the whole expression:

$$\frac{1}{4} \le \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} \le \frac{3}{4} \quad \Big|\cdot\left(x - \frac{\pi}{6}\right)$$

$$\frac{1}{4}\left(x - \frac{\pi}{6}\right) \le f(x) - 0 \le \frac{3}{4}\left(x - \frac{\pi}{6}\right)$$

$$\frac{1}{4}x - \frac{\pi}{24} \le f(x) \le \frac{3}{4}x - \frac{3\pi}{24}$$

$$\frac{1}{4}x - \frac{\pi}{24} \le f(x) \le \frac{3}{4}x - \frac{\pi}{8}$$