Calculus 1 — Exam 3
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{du}\left(e^{2u}\cos u\right) = e^{2u}\cdot 2\cdot\cos u + e^{2u}\cdot(-\sin u)$

$= e^{2u}(2\cos u - \sin u)$

2. (4pts) $\frac{d}{dx}e^{\tan x} = e^{\tan x}\cdot\sec^2 x$

3. (7pts) $\frac{d}{d\theta}\left(\sec^2\theta - \tan^2\theta\right) = 2\sec\theta\cdot\sec\theta\tan\theta - 2\tan\theta\cdot\sec^2\theta$

$= 2\sec^2\theta\tan\theta - 2\tan\theta\sec^2\theta = 0$

4. (7pts) $\frac{d}{dx}\cos\left(\sqrt[3]{x^3 - 3x^2 + 14}\right) = -\sin\left(\sqrt[3]{x^3-3x^2+14}\right)\cdot\frac{1}{3}(x^3-3x^2+14)^{-\frac{2}{3}}\cdot(3x^2-6x)$

$= \frac{-(x^2-2x)\sin\left(\sqrt[3]{x^3-3x^2+14}\right)}{(x^3-3x^2+14)^{2/3}}$

5. (8pts) $\frac{d}{d\theta}\frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta\cos^2\theta - \sin\theta\cdot 2\cos\theta(-\sin\theta)}{\cos^4\theta}$

$= \frac{\cos\theta(\cos^2\theta + 2\sin^2\theta)}{\cos^4\theta} = \frac{\cos^2\theta + \sin^2\theta + \sin^2\theta}{\cos^3\theta} = \frac{1+\sin^2\theta}{\cos^3\theta}$

6. (8pts) Use implicit differentiation to find y'.

$\sin(x+y) = x\cos y \quad \Big| \frac{d}{dx}$

$\cos(x+y)(1+y') = 1\cdot\cos y + x(-\sin y)y'$

$\cos(x+y) + y'\cos(x+y) = \cos y - xy'\sin y$

$y'\cos(x+y) + xy'\sin y = \cos y - \cos(x+y)$

$y'(\cos(x+y) + x\sin y) = \cos y - \cos(x+y)$

$y' = \frac{\cos y - \cos(x+y)}{\cos(x+y) + x\sin y}$

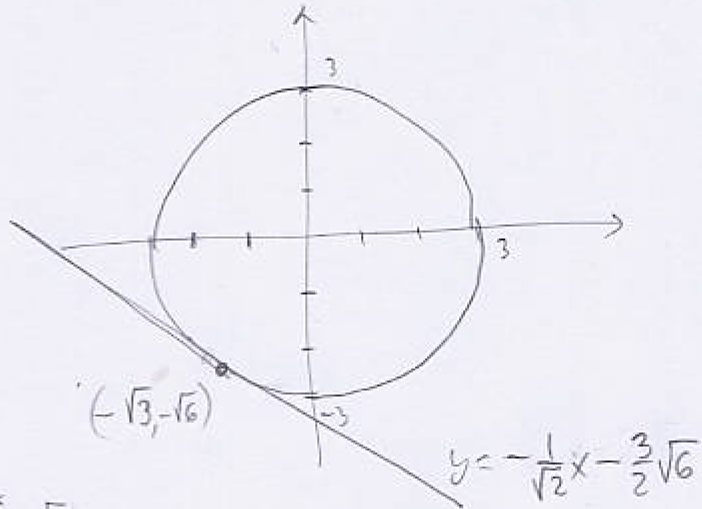
7. (10pts) The circle of radius 3 centered at the origin has equation $x^2 + y^2 = 9$. Use implicit differentiation to find the equation of the tangent line at point $(-\sqrt{3}, -\sqrt{6})$. Draw the picture of the circle and the tangent line.

$x^2+y^2=9 \quad \Big| \frac{d}{dx}$

$2x+2yy'=0$

$y' = -\frac{2y}{2x} = -\frac{x}{y}$

$y'\Big|_{(x,y)=(-\sqrt{3},-\sqrt{6})} = -\frac{-\sqrt{3}}{-\sqrt{6}} = -\frac{1}{\sqrt{2}}$



$y-(-\sqrt{6}) = -\frac{1}{\sqrt{2}}(x-(-\sqrt{3}))$

$y = -\frac{1}{\sqrt{2}}x - \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{6} = -\frac{1}{\sqrt{2}}x - \frac{\sqrt{6}}{2} - \sqrt{6}$

$= -\frac{1}{\sqrt{2}}x - \frac{3}{2}\sqrt{6}$

8. (14pts) A lemon is thrown from ground level at initial velocity 20m/s.
a) Write the formula for the position of the lemon at time t (you may assume $g \approx 10$).
b) Write the formula for the velocity of the lemon at time t.
c) What is the lemon's velocity when it is at its highest point?
d) When does the lemon reach its highest point? What is the highest point?
e) What is the acceleration when the lemon is at its highest point? How about at time $t = 0$?

a) $s(t) = 0 + 20t - \frac{10}{2}t^2$
$= 20t - 5t^2$

b) $v(t) = 20 - 10t$

c) $v(t) = 0$

d) $v(t) = 0$ when
$20 - 10t = 0$
$t = 2$

$s(2) = 20 \cdot 2 - 5 \cdot 2^2$
$= 40 - 20 = 20\,m$

e) $a(t) = -10$
$a(2) = -10$
$a(0) = -10$

9. (14pts) If you bungee-jump with a cord of length x meters, the time in seconds you will spend free-falling (that is, until the bungee cord engages) is given by $t = \frac{\sqrt{x}}{\sqrt{5}}$.

a) Find the free-falling time of a bungee cord of length 45m.

b) Find the ROC of time with respect to cord length when $x = 45$ (units?).

c) Use b) to estimate the change in time if cord length increases by 10m.

d) Use c) to estimate the free-falling time for $x = 55$m and compare to the actual value of 3.3166s.

a) $t(45) = \frac{\sqrt{45}}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}} = 3s$

b) $t' = \frac{d}{dx}\left(\frac{1}{\sqrt{5}}\sqrt{x}\right) = \frac{1}{\sqrt{5}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{5x}}$

$t'(45) = \frac{1}{2\sqrt{5 \cdot 45}} = \frac{1}{2\sqrt{5 \cdot 5 \cdot 9}} = \frac{1}{2 \cdot 5 \cdot 3} = \frac{1}{30}$ s/m

c) $\Delta t \approx t'(45) \cdot \Delta x = \frac{1}{30} \cdot 10 = \frac{1}{3}$ s

d) $t(55) \approx t(45) + \Delta t \approx 3 + \frac{1}{3} = 3.3333\ldots$, fairly close to 3.3166

10. (12pts) Let $f(x) = x^{-4}$.

a) Find the first four derivatives of f.

b) Find the general formula for $f^{(n)}(x)$.

a) $y = x^{-4}$

$y' = -4x^{-5}$

$y'' = (-4)(-5)x^{-6}$

$y''' = (-4)(-5)(-6)x^{-7}$

$y^{(4)} = (-4)(-5)(-6)(-7)x^{-8}$

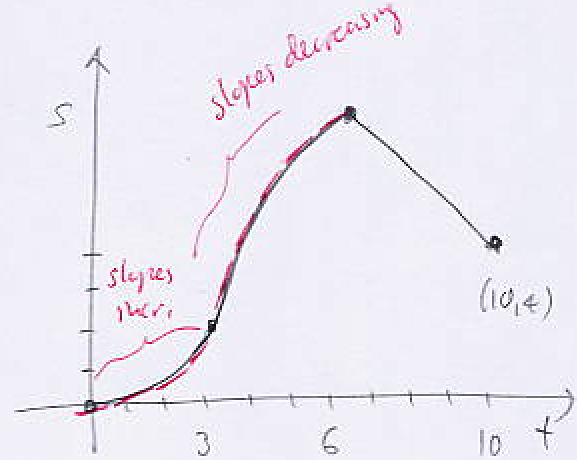
$y^{(n)} = (-4)(-5)\ldots(-(n+3))x^{-(n+4)}$

$= (-1)^n\, 4 \cdot 5 \cdot \ldots \cdot (n+3)\, x^{-(n+4)}$

$= \frac{(-1)^n\, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots (n+3)\, x^{-(n+4)}}{1 \cdot 2 \cdot 3}$

$= (-1)^n \frac{(n+3)!}{6} x^{-(n+4)}$

11. (10pts) A remote-controlled bunny's position $s(t)$ is tracked for 10 seconds. Draw the graph of its position function if we know the following:
— $s(0) = 0$, $s(10) = 4$
— it moves forward on interval $(0, 6)$
— it moves backwards on interval $(6, 10)$
— it accelerates on interval $(0, 3)$
— it decelerates on interval $(3, 6)$
— it moves at a steady velocity on interval $(6, 10)$.



Bonus. (10pts) Let $f(x) = x^2 e^x$.
a) Find the first five derivatives of f.
b) Find the pattern for $f^{(n)}(x)$.

a) $y = x^2 e^x = e^x x^2$

$y' = e^x x^2 + e^x \cdot 2x = e^x(x^2 + 2x)$

$y'' = e^x(x^2 + 2x) + e^x(2x + 2) = e^x(\underbrace{x^2 + 4x + 2}_{p})$

$y''' = e^x(\underbrace{x^2 + 4x + 2}_{p} + \underbrace{2x + 4}_{p'}) = e^x(x^2 + 6x + 6)$

$y^{(4)} = e^x(x^2 + 6x + 6 + 2x + 6) = e^x(x^2 + 8x + 12)$

$y^{(5)} = e^x(x^2 + 8x + 12 + 2x + 8) = e^x(x^2 + 10x + 20)$

0 +2, 2 +4, 6 +6, 12 +8, 20

grows with increasing speed, suggests n^2 for formula

clearly $2 \cdot n$

guess: $(n-1)^2 + (n-1)$
$= n(n-1)$

$y^{(n)} = e^x(x^2 + 2nx + n(n-1))$