Calculus 1 — Exam 2
MAT 250, Spring 2012 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(2x^7 - \frac{5}{x^3} + \sqrt[4]{x^7} + e^2\right) = 14x^6 + 15x^{-4} + \frac{7}{4}x^{\frac{3}{4}}$

$-5x^{-3} + x^{\frac{7}{4}}$ constant

2. (6pts) $\frac{d}{dt}(t^2 + yt)e^t = (2t+y)e^t + (t^2+yt)e^t = e^t(t^2+(y+2)t+y)$

3. (8pts) $\frac{d}{dx}\frac{3x-1}{x^3-5x^2+17} = \frac{3(x^3-5x^2+17)-(3x-1)(3x^2-10x)}{(x^3-5x^2+17)^2}$

$= \frac{3x^3-15x^2+51-(9x^3-33x^2+10x)}{(x^3-5x^2+17)^2} = \frac{-6x^3+18x^2-10x+51}{(x^3-5x^2+17)^2}$

4. (9pts) $\frac{d}{dw}\frac{w+\sqrt[4]{w}}{w-\sqrt[4]{w}} = \frac{(1+\frac{1}{4}w^{-\frac{3}{4}})(w-w^{\frac{1}{4}})-(w+w^{\frac{1}{4}})(1-\frac{1}{4}w^{-\frac{3}{4}})}{(w-w^{1/4})^2}$ $\quad -\frac{3}{2}w^{1/4}$

$= \frac{\cancel{w}+\frac{1}{4}w^{\frac{1}{4}}-w^{\frac{1}{4}}-\cancel{\frac{1}{4}w^{-\frac{1}{2}}}-(\cancel{w}+w^{\frac{1}{4}}-\frac{1}{4}w^{\frac{1}{4}}-\cancel{\frac{1}{4}w^{-\frac{1}{2}}})}{(w-w^{1/4})^2} = \frac{-\frac{3}{4}w^{\frac{1}{4}}-(\frac{3}{4}w^{\frac{1}{4}})}{(w-w^{\frac{1}{4}})^2}$

$= -\frac{3w^{1/4}}{2(w-w^{1/4})^2}$

5. (6pts) Let $h(x) = \frac{f(x)+g(x)}{f(x)g(x)}$. Find the general expression for $h'(x)$ and simplify.

$h'(x) = \frac{(f'(x)+g'(x))(f(x)g(x))-(f(x)+g(x))(f'(x)g(x)+f(x)g'(x))}{(f(x)g(x))^2}$

$= f'(x)\cancel{f(x)}g(x)+f(x)\cancel{g(x)}g'(x)-(f(x)\cancel{f'(x)}g(x)+f'(x)g(x)^2+f(x)^2g'(x)+f(x)\cancel{g(x)}g(x))$

$= \frac{-f'(x)g(x)^2-f(x)^2g'(x)}{(f(x)g(x))^2}$

Find the following limits algebraically.

6. (5pts) $\lim_{x\to 3} \dfrac{x^2+x-12}{x^2-10x+21} = \lim_{x\to 3} \dfrac{\cancel{(x-3)}(x+4)}{\cancel{(x-3)}(x-7)} = \dfrac{3+4}{3-7} = -\dfrac{7}{4}$

7. (7pts) $\lim_{x\to 25} \dfrac{5-\sqrt{x}}{25-x} \cdot \dfrac{5+\sqrt{x}}{5+\sqrt{x}} = \lim_{x\to 25} \dfrac{\cancel{25-x}}{\cancel{(25-x)}(5+\sqrt{x})} = \dfrac{1}{5+\sqrt{25}} = \dfrac{1}{10}$

8. (7pts) $\lim_{x\to 0} \dfrac{\sin(3x)}{x^2-x} = \lim_{x\to 0} \dfrac{\sin(3x)}{x(x-1)} \cdot \dfrac{3}{3} = \lim_{x\to 0} \underbrace{\dfrac{\sin(3x)}{3x}}_{\to 1} \cdot \dfrac{3}{x-1} = 1 \cdot \dfrac{3}{0-1} = -3$

9. (10pts) Find $\lim_{x\to 0^+} x^3\left(4+\sin^2\left(\dfrac{1}{x}\right)\right)$. Use the theorem that rhymes with what unkind children do to their peers.

$-1 \le \sin\frac{1}{x} \le 1$

So $0 \le \sin^2\frac{1}{x} \le 1$ $\quad 1+4$

$4 \le 4+\sin^2\frac{1}{x} \le 5$

$4x^3 \le x^3\left(4+\sin^2\frac{1}{x}\right) \le 5x^3$

$\lim_{x\to 0+} 4x^3 = 0$

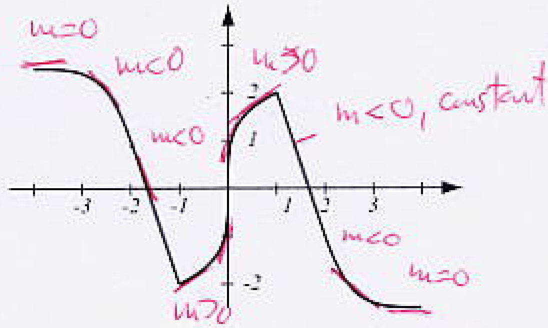
$\lim_{x\to 0+} 5x^3 = 0$

Are equal, so by squeeze theorem

$\lim_{x\to 0+} x^3\left(4+\sin^2\left(\frac{1}{x}\right)\right) = 0$

10. (12pts) The graph of the function $f(x)$ is shown at right.

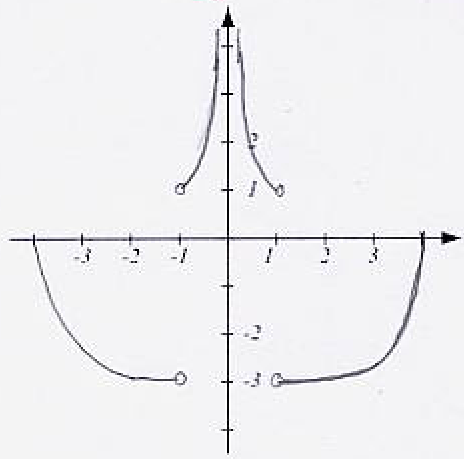
a) Find the points where $f'(a)$ does not exist.
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?



a) $x = -1, 1, 0$

sharp point ← vertical tangent line

b) picture



c) f is odd

f' is even.

11. (16pts) Let $f(x) = \dfrac{x}{x+1}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(1, \frac{1}{2})$.

a) $$f'(a) = \lim_{x \to a} \frac{\frac{x}{x+1} - \frac{a}{a+1}}{x-a} = \lim_{x \to a} \frac{\frac{x(a+1) - a(x+1)}{(x+1)(a+1)}}{x-a}$$

$$= \lim_{x \to a} \frac{\cancel{xa} + x - \cancel{ax} - a}{(x+1)(a+1)} \cdot \frac{1}{x-a} = \lim_{x \to a} \frac{\cancel{x-a}}{(x+1)(a+1)(\cancel{x-a})} = \frac{1}{(a+1)^2}$$

so $f'(x) = \dfrac{1}{(x+1)^2}$

b) $$\left(\frac{x}{x+1}\right)' = \frac{1 \cdot (x+1) - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

c) $y'(1) = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$

$y - \frac{1}{2} = \frac{1}{4}(x-1)$

$y = \frac{1}{4}x - \frac{1}{4} + \frac{1}{2} = \frac{1}{4}x + \frac{1}{4}$

12. (8pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Use the infomation above and differentiation formulas to find the limit.

$$\lim_{x\to 32} \frac{\sqrt[5]{x}-2}{x-32} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

a) $a = 32$

$f(x) = \sqrt[5]{x}$

$f(a) = \sqrt[5]{32} = 2$

b) $f'(32) = \frac{d}{dx} x^{\frac{1}{5}} \Big|_{x=32} = \frac{1}{5} x^{-\frac{4}{5}} \Big|_{x=32}$

$= \frac{1}{5} \cdot \frac{1}{(\sqrt[5]{32})^4} = \frac{1}{5 \cdot 16} = \frac{1}{80}$

Bonus. (10pts) We have indicated how to prove $(x^n)' = nx^{n-1}$ for $n \geq 0$. Show that the formula works for integers $n < 0$ as follows: set $n = -k$, and develop the rule for the derivative of x^{-k} with the help of the quotient rule and the rule for positive exponents.

$y = x^{-k} = \frac{1}{x^k}$ $\quad k \geq 0 \quad (n = -k)$

$$y' = \frac{0 \cdot x^k - 1 \cdot kx^{k-1}}{(x^k)^2} = -\frac{kx^{k-1}}{x^{2k}} = -kx^{k-1-2k} = \underbrace{-k}_{n}\, x^{\overbrace{-k-1}^{n}}$$

Putting back $n = -k$, we get $(x^n)' = nx^{n-1}$