Calculus 1 — Exam 1
MAT 250, Spring 2011 — D. Ivanšić

Name: ____________________

Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -2^-} f(x) =$

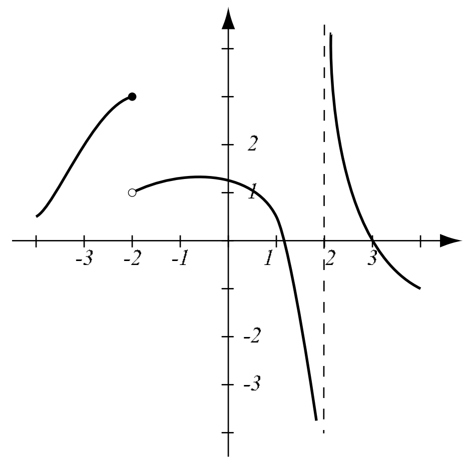
$\lim_{x\to -2^+} f(x) =$

$\lim_{x\to -2} f(x) =$

$\lim_{x\to 2^-} f(x) =$

$\lim_{x\to 2^+} f(x) =$

$\lim_{x\to 2} f(x) =$



List points where f is not continuous and justify why it is not continuous at those points.

2. (4pts) Find the following limit algebraically (no need to justify, other than showing the computation).

$\lim_{x\to 2} (x^2 + 4x - 6) =$

3. (8pts) Let $\lim_{x\to 4} f(x) = 1$ and $\lim_{x\to 4} g(x) = -2$. Use limit laws to find the limit below and show each step.

$$\lim_{x\to 4} \frac{x^2 - 2g(x)}{2f(x) - g(x) + 7} =$$

4. (12pts) Let $f(x) = \dfrac{\sin x}{x}$.
a) Find the domain of f.
b) Explain, using continuity laws, why the function is continuous on its domain.
c) At points of discontinuity, state the type of discontinuity (jump, infinite, removable) and explain. Use a well-known limit that we mentioned for this.

5. (16pts) The height of a blackberry (fruit or phone — your choice!) t seconds after getting thrown upwards with initial velocity 30 meters per second is given by $h(t) = 30t - 5t^2$ (in meters).
a) Find the average velocities of the blackberry over six short intervals of time, three of them beginning with 2, and three ending with 2. Show the table of values. What are the units?
b) Use the information in a) to find the instantaneous velocity of the blackberry at $t = 2$. What are the units?

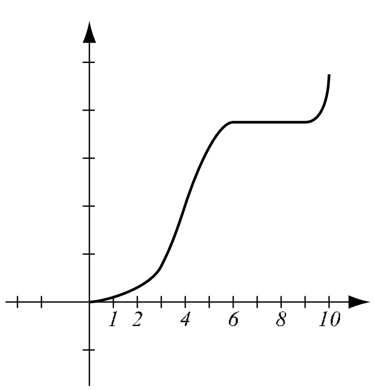
6. (10pts) Find the following limit algebraically (do not use the calculator) and justify.

$$\lim_{x\to 3+} \frac{2x-7}{x-3} =$$

7. (18pts) Let $f(x) = \sqrt{x}$, and let $P = (9, 3)$.
a) Draw the graph of f on the interval $[0, 11]$.
b) Draw three secant lines PQ, where Q is to the left of P.
c) If $Q = (x, f(x))$ is a general point on the graph of f, write the formula for the slope of the secant line PQ.
d) Find slopes of three secant lines where Q is close to P (show table) and use those slopes to find the slope of the tangent line at P.

8. (16pts) Consider the limit $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.
a) Can you use a limit law to find the limit? Why or why not?
b) Use your calculator to estimate the limit — write down the table on paper. Make a guess as to what the limit is exactly.
b) What does the calculator give you if you take an x very close to 0? Does this alter your estimate of the limit? Why or why not?

Bonus. (10pts) Below is the graph of the position of a car t minutes after noon. Answer the following, with justification.
a) Is there a time interval when the car is not moving? If so, when?
b) When is the car speeding up? Give time intervals.
c) When is the car slowing down? Give time intervals.



Calculus 1 — Exam 2
MAT 250, Spring 2011 — D. Ivanšić

Name: ____________________

Show all your work!

1. (17pts) Differentiate and simplify where appropriate:

$$\frac{d}{dx}\left(4x^3 - \frac{3}{x^6} + \sqrt[3]{x^{13}} + 4^5\right) =$$

$$\frac{d}{dx}\left((\sqrt{x}+7)(3\sqrt[4]{x} - x^2)\right) =$$

$$\frac{d}{dy}\left(4e^y + 3e^4\right) =$$

2. (10pts) Use the Intermediate Value Theorem to show that the equation $\cos x = x - 1$ has at least one solution. Write a nice sentence that shows how you are using the IVT.

3. (22pts) Find the following limits algebraically.

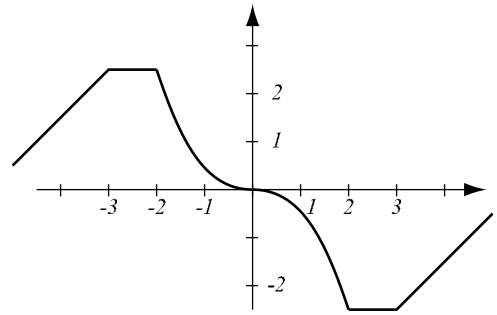
$$\lim_{x \to -4} \frac{x^2 - 3x - 28}{x^2 - 16} =$$

$$\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1} =$$

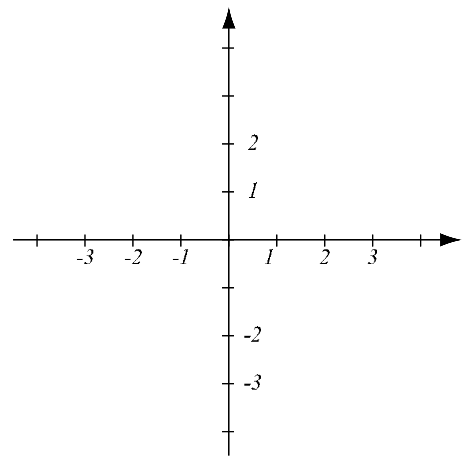
$$\lim_{x \to 0} \frac{\sin(7x)}{\sin(5x)} =$$

4. (10pts) Find $\lim_{x \to 0} x^2 \left(\sin \frac{1}{x} + \cos \frac{1}{x}\right)$. Use the theorem that rhymes with what you pay at the bursar's office, other than tuition.

5. (15pts) The graph of the function $f(x)$ is shown at right.



a) Find the points where $f'(a)$ does not exist.
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?



6. (16pts) Let $f(x) = \dfrac{1}{x^2}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(2, \frac{1}{4})$.

7. (10pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Use the infomation above to find the limit.

$$\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h}$$

Bonus. (10pts) Use the limit definition of the derivative to find the derivative of the function $f(x) = \dfrac{x^2}{x+3}$.

Calculus 1 — Exam 3 **MAT 250, Spring 2011 — D. Ivanšić**

Name: ____________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left((x^4 - 4x)\sin x\right) =$

2. (8pts) $\dfrac{d}{dx}\,\dfrac{5x+3}{x^2-7x+4} =$

3. (8pts) $\dfrac{d}{dz}\,\dfrac{z+\sqrt{z}}{z-\sqrt{z}} =$

4. (8pts) $\dfrac{d}{d\theta}\,\dfrac{\cos\theta - 1}{\sin\theta} =$

5. (8pts) Let $f(3) = 2$, $f'(3) = -1$, $g(3) = 4$ and $g'(3) = -2$, and let $h(x) = \dfrac{xf(x)}{g(x)}$.

a) Find the general expression for $h'(x)$.

b) Find $h'(3)$.

6. (10pts) Find the equation of the tangent line to the curve $y = \tan^2 x$ at the point $x = \frac{\pi}{4}$.

7. (16pts) A pomegranate is thrown upwards so that at height 30m it has upward velocity 10m/s.

a) Write the formula for the position of the pomegranate at time t (you may assume $g \approx 10$ and take $t = 0$ to be the time of the above observation).

b) When does the pomegranate reach height 15m on the way down? On the way up?

c) Write the formula for the velocity of the pomegranate at time t.

d) What are the velocities of the pomegranate at the times from b)?

8. (14pts) The volume in cm^3 of a cantaloup is given by the formula $V = \frac{1}{10}A^{\frac{3}{2}}$, where A is its surface area in cm^2.
a) Find the volume of a cantaloup whose surface area is 900cm^2.
b) Find the ROC of volume with respect to surface area when $A = 900$ (units?).
c) Use b) to estimate the change in volume if surface area decreases by 50cm^2.
d) Use c) to estimate the volume of a cantaloup with surface area 850cm^2 and compare to the actual value of 2478.1546cm^3.

9. (12pts) Let $f(x) = x^{-1}$.
a) Find the first four derivatives of f.
b) Find the general formula for $f^{(n)}(x)$.

10. (10pts) An automobile's position is tracked for 10 seconds. Draw the graph of its position function if we know the following:
— it is always moving forward
— it accelerates on interval $(0, 4)$
— it moves at a steady velocity on interval $(4, 7)$
— it decelerates on interval $(7, 9)$
— it is at rest on interval $(9, 10)$.

Bonus. (10pts) Let $f(x) = e^x \sin x$.
a) Find the first four derivatives of f.
b) Find the pattern for $f^{(n)}(x)$. (You may need to describe it in words.)
c) What is $f^{(35)}(x)$?

Calculus 1 — Exam 4
MAT 250, Spring 2011 — D. Ivanšić

Name: ____________________

Show all your work!

Differentiate and simplify where appropriate:

1. (4pts) $\dfrac{d}{dx}\sqrt{x^3-4x^2+1}=$

2. (6pts) $\dfrac{d}{dx}\,e^{3x}\sin 5x=$

3. (7pts) $\dfrac{d}{du}\,e^{\frac{1}{u^2+5u}}=$

4. (8pts) $\dfrac{d}{dx}\,\ln\dfrac{x^2+1}{(x+4)^2}=$

5. (6pts) $\dfrac{d}{dy}\,7^{7^y}=$

6. (9pts) $\dfrac{d}{dx}\,\dfrac{\arccos x}{\sqrt{1-x^2}}=$

7. (12pts) Use implicit differentiation to find the equation of the tangent line to the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ at the point $(\sqrt{5}, \frac{4}{\sqrt{5}})$. Draw the picture of the ellipse and the tangent line.

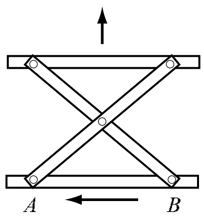
8. (10pts) Use implicit differentiation to find y'.

$x^4 + y^4 = \dfrac{x}{y}$

9. (10pts) Use logarithmic differentiation to find the derivative of $y = (\arctan x)^x$.

10. (12pts) Let $f(x) = x^2 - 4x$, $x \geq 2$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find $g'(12)$.

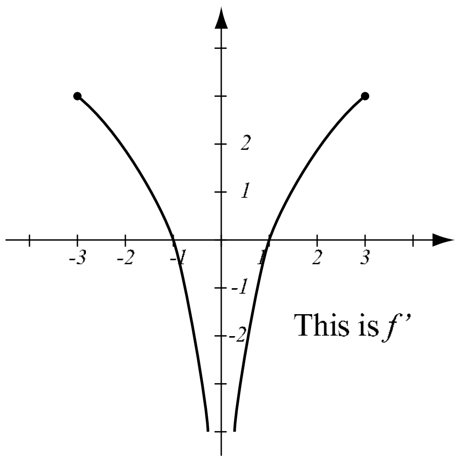
11. (16pts) A workers' platform raises as point B is pulled by a chain towards the stationary point A (circles denote rotating joints). If the diagonal beams are 5 meters long, and B is pulled towards A at rate 2 meters per minute, at what rate is the top of the platform rising when B is 3 meters away from A? What are the units? *(Hint: use a triangle.)*

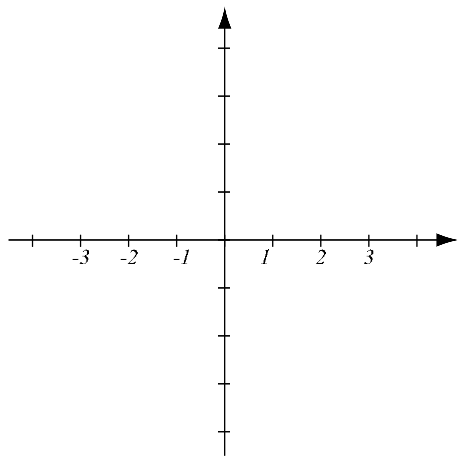


Bonus. (10pts) In the previous problem let θ be the angle between either of the diagonal beams and the horizontal. Find how fast θ is increasing at the moment described above.

Calculus 1 — Exam 5	**Name:** ______________________
MAT 250, Spring 2011 — D. Ivanšić	*Show all your work!*

1. (18pts) Let f be continuous on $[-3, 3]$. The graph of its derivative f' is drawn below. Note that f' is not defined at 0. Use the graph to answer:
a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?
b) What are the intervals of concavity of f? Where does f have inflection points?
c) Use the information gathered in a) and b) to draw one possible graph of f at right. Note that f *is* defined at 0, make it, for example, $f(0) = 1$.





2. (12pts) Use Rolle's Theorem to show that the equation $2x + \sin x = 0$ has at most one solution.

3. (12pts) Verify the Mean Value Theorem for the function $f(x) = \ln x$ on the interval $[1, e^2]$. (Approximate $e \approx 3$ when necessary).

4. (22pts) Let $f(x) = \sin^4 x + \cos^4 x$, where $x \in [0, \pi]$.
a) Find the critical points and the intervals of increase and decrease of f. Determine where f has a local maximum or minimum.
b) Using information found in a), draw a rough sketch of f.

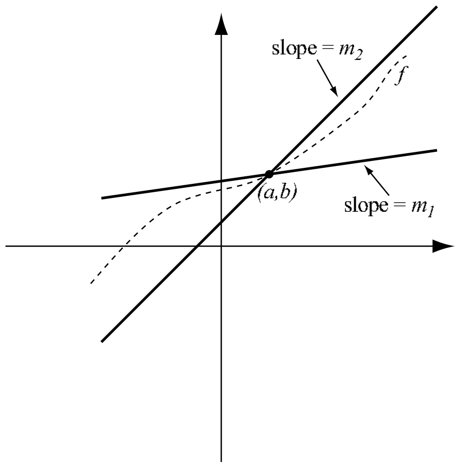
5. (14pts) Let $f(x) = \dfrac{x+3}{x^2+7}$. Find the absolute minimum and maximum values of f on the interval $[-1, 2]$.

6. (22pts) Let $f(x) = x^2 e^x$.

a) Find the intervals of concavity and points of inflection for f.

b) Find the critical points and use a) to determine which are local minima or maxima.

Bonus. (10pts) Let f be differentiable for all real numbers x so that $m_1 \leq f'(x) \leq m_2$ and $f(a) = b$. Use the Mean Value Theorem to show that the graph of f must be between the two lines with slopes m_1 and m_2, shown in the picture.



Calculus 1 — Exam 6
MAT 250, Spring 2011 — D. Ivanšić

Name: ______________________

Show all your work!

Find the limits. Use L'Hopital's rule where appropriate.

1. (10pts) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt[10]{x}} =$

Which grows faster, $\ln x$ or $\sqrt[10]{x}$?

2. (8pts) $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2} =$

3. (12pts) $\lim_{x\to 0+} (\sin x)^x =$

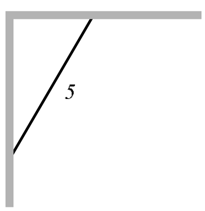
4. (8pts) $\lim_{x\to\infty} \sqrt{x^4+1} =$

5. (32pts) Let $f(x) = \dfrac{x^2}{x^2 - 4}$. Draw an accurate graph of f by following the guidelines.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
d) Use information from a)–d) to sketch the graph.

6. (8pts) Find the limit without using L'Hopital's rule.

$$\lim_{x\to\infty} \frac{5x^3+3x^2-x}{-x^2+7x-4} =$$

7. (22pts) Jodie has a thick wooden pole 5 meters long that she will lay on the ground as a boundary for a triangular flower plot along the walls at a corner of her house (see picture).
a) Draw two more options for positioning the pole.
b) How should she position the pole so that the enclosed area is maximal? (Don't forget to check that it is a maximum.)



Bonus. (10pts) Show that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$. Note that this knowledge allows you to compute e approximately by putting a large x into the expression $\left(1+\frac{1}{x}\right)^x$ (although this is not a very efficient way to do it).

Calculus 1 — Exam 7 MAT 250, Spring 2011 — D. Ivanšić

Name: ______________________________

Show all your work!

Find the following antiderivatives.

1. (4pts) $\displaystyle\int \cos\left(5x - \frac{\pi}{2}\right)\, dx =$

2. (7pts) $\displaystyle\int (x^3 - 4x)\sqrt{x}\, dx =$

3. (5pts) $\displaystyle\int \frac{1}{1 + (3x)^2},\, dx =$

4. (16pts) Find $\int_{-1}^{3} |x|\, dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Fundamental Theorem of Calculus (you will have to break it up into two integrals).

5. (6pts) Write in sigma notation.

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} =$$

Use the substitution rule in the following integrals:

6. (9pts) $\displaystyle\int \frac{4x^3 + 14x}{\sqrt[5]{x^4 + 7x^2 + 9}}\, dx =$

7. (10pts) $\displaystyle\int_{\ln 3}^{\ln 5} \frac{e^x}{(7 + e^x)^2}\, dx =$

8. (6pts) $\displaystyle\int_0^4 \frac{x - 2}{\sin(x^2 - 4x + 7)}\, dx =$

9. (8pts) A rocket shoots up vertically with velocity $v(t) = 5t^3 + 4t^2$ (in meters/second). Find its position function $s(t)$, if at time $t = 6$, the rocket is at altitude 1800m.

10. (21pts) The function $f(x) = e^x$, $0 \le x \le 3$ is given.
a) Write down the expression that is used to compute M_6. Then compute M_6.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does M_6 represent?
c) Using the Fundamental Theorem of Calculus, evaluate $\int_0^3 e^x \, dx$. Is M_6 an overestimate or an underestimate of this integral?

11. (8pts) Show that $\frac{\pi}{3} \leq \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x \, dx \leq \frac{2\pi}{3}$ **without** evaluating the integral.

Bonus. (10pts) The rate at which money flows in or out of a company's account is given by the formula $3t^2 - 48$ dollars/day, $0 \leq t \leq 30$, t in days. At time $t = 0$, there was \$800 in the account.

a) When is the company losing money from the account, and when is it gaining money?

b) By how much did the amount in the account increase or decrease from $t = 0$ to $t = 6$?

c) How much money is in the account when $t = 6$?

Calculus 1 — Final Exam **MAT 250, Spring 2011 — D. Ivanšić**	**Name:** ______________ *Show all your work!*

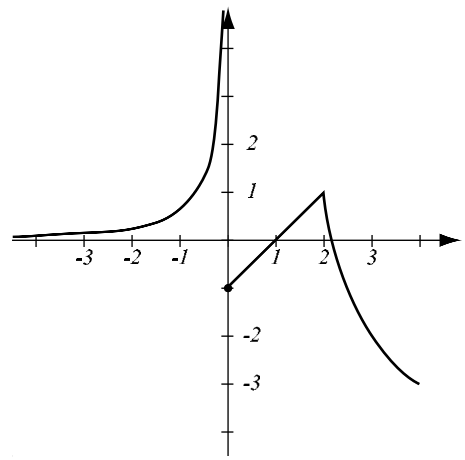
1. (14pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x \to 0^-} f(x) =$

$\lim_{x \to 0^+} f(x) =$

$\lim_{x \to 0} f(x) =$

$\lim_{x \to -\infty} f(x) =$



List points where f is not continuous and explain why.

List points where f is not differentiable and explain why.

2. (12pts) Find $\lim_{x \to \infty} \dfrac{x^2 - 3x + 4}{x^2 - 9}$ in two ways: a) algebraically b) Using L'Hospital's rule.

3. (10pts) Find the absolute minimum and maximum values for the function $f(x) = x + 2\sin x$ on the interval $[0, \pi]$.

4. (10pts) Use implicit differentiation to find y'.

$$y \ln x = y^2 + \frac{x}{y}$$

5. (14pts) Let $V(t) = t \arctan t$ be the volume, measured in liters, of water in a tank at time t minutes.

a) What is the volume at $t = 1$?

b) Find $V'(1)$. What does it represent? What are the units?

c) Use the numbers from a) and b) to approximate the volume at time $t = 1.3$.

d) What is the exact volume at time $t = 1.3$?

6. (21pts) Let $f(x) = x^3 e^x$.
a) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. (Use L'Hopital's rule if necessary.)
b) Find the intervals of increase/decrease and where f has a local maximum and minimum.
c) Find the intervals where f is concave up or down and where it has an inflection point.
d) Use the results of a), b) and c) to accurately sketch the graph of f.

7. (10pts) Use logarithmic differentiation to find the derivative of $y = x^x$.

8. (13pts) At time $t = 0$ a car starts moving westward from an intersection at speed 45mph. At time $t = 1$hr another car starts moving southward from the same intersection, with speed 60mph. At what rate are the cars moving apart at time $t = 2$hrs?

9. (12pts) Consider the integral $\displaystyle\int_1^3 x^2 - 6x + 8\,dx$.

a) Use a picture to determine whether this definite integral is positive or negative.

b) Evaluate the integral and verify your conclusion from a).

10. (6pts) Find $f(x)$ if $f'(x) = \dfrac{x^3+4}{x^2}$ and $f(2) = -3$.

11. (6pts) Find the indefinite integral:

$$\int e^{7x-3} + \sqrt[4]{x^7}\, dx =$$

12. (10pts) Use substitution to evaluate:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx =$$

13. (12pts) The equation $e^x + x = 2$ is given.
a) Use the Intermediate Value Theorem to show that this equation has at least one real solution.
b) Use Rolle's Theorem to show it cannot have more than one solution.

Bonus. (14pts) Find the point on the curve $y = \frac{1}{x^2}$, $x > 0$, that is closest to the origin. Show that the point you find is, indeed, the closest.