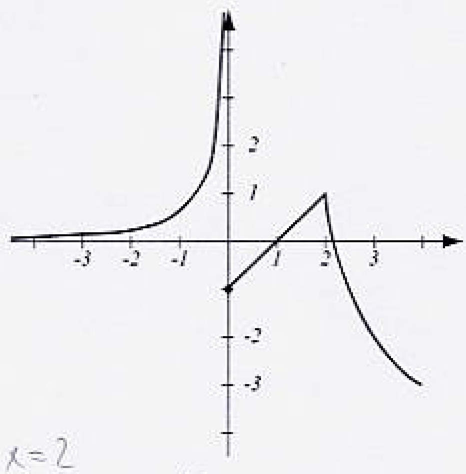
Calculus 1 — Final Exam   **MAT 250, Spring 2011 — D. Ivanšić**

Name: Solution

Show all your work!

1. (14pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.



$\lim_{x\to 0^-} f(x) = \infty$

$\lim_{x\to 0^+} f(x) = -1$

$\lim_{x\to 0} f(x) =$ d.n.e. one-sided limits are not equal

$\lim_{x\to -\infty} f(x) = 0$

List points where f is not continuous and explain why.

f not cont. at $x=0$ because $\lim_{x\to 0} f(x)$ d.n.e.

List points where f is not differentiable and explain why.

f is not diff. at $x=0$ (not even cont. there) and $x=2$ (sharp point)

2. (12pts) Find $\lim_{x\to\infty} \dfrac{x^2-3x+4}{x^2-9}$ in two ways: a) algebraically b) Using L'Hospital's rule.

a) $\lim_{x\to\infty} \dfrac{x^2-3x+4}{x^2-9} = \lim_{x\to\infty} \dfrac{x^2\left(1-\frac{3}{x}+\frac{4}{x^2}\right)}{x^2\left(1-\frac{9}{x^2}\right)} = \dfrac{1-0+0}{1-0} = 1$

b) $\lim_{x\to\infty} \dfrac{\overbrace{x^2-3x+4}^{\to\infty}}{\underbrace{x^2-9}_{\to\infty}} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{\overset{\to\infty}{2x-3}}{\underset{\to\infty}{2x}} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{2}{2} = 1$

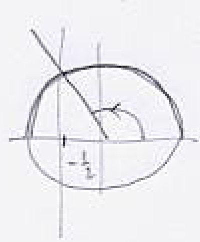
3. (10pts) Find the absolute minimum and maximum values for the function $f(x) = x + 2\sin x$ on the interval $[0, \pi]$.

$f'(x) = 1 + 2\cos x$

$f'(x) = 0$

$1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$



$x = \frac{2\pi}{3}$

x	$x+2\sin x$		
0	0		min.
π	$\pi+0$	≈ 3.14	
$\frac{2\pi}{3}$	$\frac{2\pi}{3}+2\cdot\frac{\sqrt{3}}{2}$	≈ 3.83	max

4. (10pts) Use implicit differentiation to find y'.

$y \ln x = y^2 + \dfrac{x}{y}$ $\quad \Big| \cdot y$

$y^2 \ln x = y^3 + x \quad \Big| \frac{d}{dx}$

$2yy' \ln x + y^2 \cdot \frac{1}{x} = 3y^2 y' + 1$

$2yy' \ln x - 3y^2 y' = 1 - \frac{y^2}{x}$

$y'(2 \cdot y \ln x - 3y^2) = 1 - \frac{y^2}{x}$

$y' = \dfrac{1 - \frac{y^2}{x}}{2y \ln x - 3y^2} = \dfrac{x - y^2}{2xy \ln x - 3xy^2}$

5. (14pts) Let $V(t) = t \arctan t$ be the volume, measured in liters, of water in a tank at time t minutes.

a) What is the volume at $t = 1$?

b) Find $V'(1)$. What does it represent? What are the units?

c) Use the numbers from a) and b) to approximate the volume at time $t = 1.3$.

d) What is the exact volume at time $t = 1.3$?

a) $V(1) = 1 \cdot \arctan 1 = \frac{\pi}{4} \approx 0.785398$

b) $V'(t) = \arctan t + t \cdot \frac{1}{1+t^2}$

$V'(1) = \frac{\pi}{4} + \frac{1}{1+1} = \frac{\pi}{4} + \frac{1}{2} \approx 1.285398$ liters/minute

(represents rate of chg of volume, or flow)

c) $\Delta V \approx V'(1) \cdot \Delta t$

$\approx \left(\frac{\pi}{4} + \frac{1}{2}\right) \cdot 0.3 \approx 0.385619$

$V(1.3) \approx V(1) + \Delta V = 0.785398 + 0.385619$

≈ 1.171018

d) $V(1.3) \approx 1.3 \cdot \arctan 1.3 = 1.189631$

6. (21pts) Let $f(x) = x^3 e^x$.

a) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. (Use L'Hopital's rule if necessary.)

b) Find the intervals of increase/decrease and where f has a local maximum and minimum.

c) Find the intervals where f is concave up or down and where it has an inflection point.

d) Use the results of a), b) and c) to accurately sketch the graph of f.

a) $\lim_{x\to\infty} x^3 \cdot e^x = \infty \cdot \infty = \infty$

$\lim_{x\to-\infty} x^3 e^x = \lim_{x\to-\infty} \frac{x^3}{e^{-x}} \overset{LH}{=} \lim_{x\to-\infty} \frac{3x^2}{-e^{-x}} = \lim_{x\to-\infty} \frac{6x}{e^{-x}} = \lim_{x\to-\infty} \frac{6}{-e^{-x}} = \frac{6}{\infty} = 0$

$-\infty \cdot 0$

b) $f'(x) = 3x^2 e^x + x^3 e^x$

$= (x^3 + 3x^2) e^x$

$(x^3+3x^2)e^x = 0$

$x^2(x+3) = 0$ or $e^x = 0$

$x = 0, -3$ no sol.

$e^x > 0$, $x^2 \geq 0$ so sign of $x^2(x+3)$ depends only on $x+3$.

		-3		0	
f'	$-$	0	$+$	0	$+$
f	$\searrow$	loc. min	$\nearrow$		$\nearrow$

c) $f''(x) = (3x^2+6x)e^x + (x^3+3x^2)e^x$

$= (x^3 + 6x^2 + 6x) e^x$

$x(x^2+6x+6)e^x = 0$

$x = 0$ $x^2+6x+6=0$ $e^x = 0$ no sol.

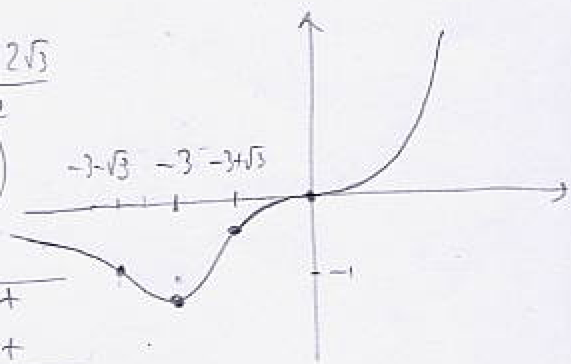
$x = \frac{-6 \pm \sqrt{36-4\cdot6\cdot1}}{2}$

$= \frac{-6 \pm \sqrt{12}}{2} = \frac{-6 \pm 2\sqrt{3}}{2}$

$= -3 \pm \sqrt{3}$ (both <0)

		$-3-\sqrt{3}$		$-3+\sqrt{3}$		0	
x	$-$		$-$		$-$	0	$+$
x^2+6x+6	$+$	0	$-$	0	$+$		$+$
f''	$-$	0	$+$	0	$-$	0	$+$
f	CD	IP	CU	IP	CD	IP	CU

x	$f(x)$
0	0
$-3-\sqrt{3}$	-0.93
$-3+\sqrt{3}$	-0.57
-3	-1.34



7. (10pts) Use logarithmic differentiation to find the derivative of $y = x^x$.

$y = x^x$

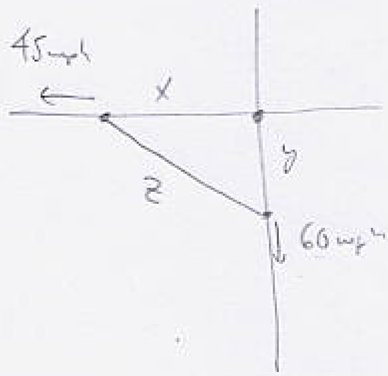
$\ln y = x \ln x \quad \Big| \frac{d}{dx}$

$\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$y' = (\ln x + 1) y$

$y' = x^x (\ln x + 1)$

8. (13pts) At time $t = 0$ a car starts moving westward from an intersection at speed 45mph. At time $t = 1$hr another car starts moving southward from the same intersection, with speed 60mph. At what rate are the cars moving apart at time $t = 2$hrs?



Know: $\frac{dx}{dt} = 45$ mph

$\frac{dy}{dt} = 60$ mph

Need: $\frac{dz}{dt}$ when $t=2$

When $t=2$, $x = 45\cdot 2 = 90$

$y = 60\cdot 1 = 60$

$z^2 = 60^2 + 90^2 = 3600 + 8100 = 11700$

$z = 10\sqrt{117} = 10\sqrt{9\cdot 13} = 30\sqrt{13}$

$x^2 + y^2 = z^2 \quad \Big| \frac{d}{dt}$

$2xx' + 2yy' = 2zz'$

$z' = \frac{xx' + yy'}{z}$

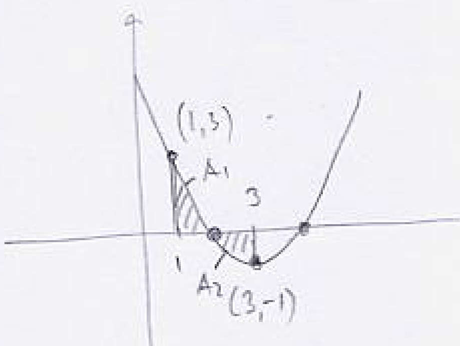
$z' = \frac{90\cdot 45 + 60\cdot 60}{30\sqrt{13}} = \frac{30(3\cdot 45 + 2\cdot 60)}{30\sqrt{13}}$

$= \frac{255}{\sqrt{13}} = 70.724275$ mph

9. (12pts) Consider the integral $\int_1^3 x^2 - 6x + 8\, dx$.

a) Use a picture to determine whether this definite integral is positive or negative.

b) Evaluate the integral and verify your conclusion from a).



a) $\int_1^3 x^2 - 6x + 8\, dx = A_1 - A_2 > 0$, since, it appears that $A_1 > A_2$.

b) $\int_1^3 x^2 - 6x + 8\, dx = \left(\frac{x^3}{3} - 6\cdot\frac{x^2}{2} + 8x\right)\Big|_1^3$

$= \frac{1}{3}(27-1) - 3(9-1) + 8(3-1) = \frac{26}{3} - 24 + 16$

$= \frac{26}{3} - 8 = \frac{26-24}{3} = \frac{2}{3} > 0$

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$3^2 - 18 + 8 = -1$

10. (6pts) Find $f(x)$ if $f'(x) = \dfrac{x^3+4}{x^2}$ and $f(2) = -3$.

$f'(x) = \frac{x^3+4}{x^2} = x + \frac{4}{x^2} = x + 4x^{-2}$

$f(x) = \frac{x^2}{2} + 4\frac{x^{-1}}{-1} = \frac{x^2}{2} - \frac{4}{x} + C$

$-3 = f(2) = \underbrace{\frac{2^2}{2} - \frac{4}{2}}_{=0} + C$

$C = -3$

$f(x) = \frac{x^2}{2} - \frac{4}{x} - 3$

11. (6pts) Find the indefinite integral:

$\int e^{7x-3} + \sqrt[4]{x^7}\,dx = \frac{e^{7x-3}}{7} + \frac{4}{11}x^{\frac{11}{4}} + C$

$x^{\frac{7}{4}}$

12. (10pts) Use substitution to evaluate:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^7 x\,dx = \begin{bmatrix} u = \tan x & x = \frac{\pi}{3},\ u = \sqrt{3} \\ du = \sec^2 x\,dx & x = \frac{\pi}{6},\ u = \frac{1}{\sqrt{3}} \end{bmatrix} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^7\,du$

$= \frac{u^8}{8}\Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{1}{8}\left(\sqrt{3}^8 - \frac{1}{\sqrt{3}^8}\right) = \frac{1}{8}\left(81 - \frac{1}{81}\right) = \frac{81^2-1}{8\cdot 81}$

$= \frac{6560}{8\cdot 81} = \frac{820}{81}$

13. (12pts) The equation $e^x + x = 2$ is given.

a) Use the Intermediate Value Theorem to show that this equation has at least one real solution.

b) Use Rolle's Theorem to show it cannot have more than one solution.

a) Let $f(x) = e^x + x - 2$

$f(0) = 1 + 0 - 2 = -1 \quad < 0$

$f(1) = e + 1 - 2 = e - 1 \approx 1.71 > 0$

Since $f(0) < 0 < f(1)$, by intermediate value theorem, there is a number c so that $f(c) = 0$

b) Suppose there are two solutions, a & b.

Then $f(a) = f(b) = 0$, By Rolle's theorem, there is a number $c \in (a, b)$ s.t. $f'(c) = 0$

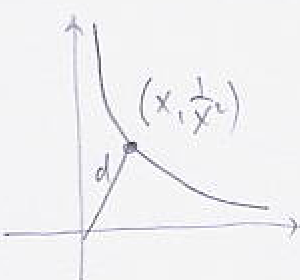
But: $f'(c) = e^x + 1x$

$e^x + 1 = 0$

$e^x = -1$ has no solution

We arrive at a contradiction, so there cannot be more than 1 solution

Bonus. (14pts) Find the point on the curve $y = \frac{1}{x^2}$, $x > 0$, that is closest to the origin. Show that the point you find is, indeed, the closest.



$d = \sqrt{(x-0)^2 + \left(\frac{1}{x^2} - 0\right)^2}$

$f(x) = d^2 = x^2 + \frac{1}{x^4}$

Job: minimize $f(x)$ on $(0, \infty)$

$f'(x) = 2x - 4x^{-5}$

$2x - \frac{4}{x^5} = 0 \quad | \cdot x^5$

$2x^6 - 4 = 0$

$x^6 = 2, \quad x = \pm\sqrt[6]{2} = \sqrt[6]{2}$ since $x > 0$

$f''(x) = 2 + 20x^{-6} = 2 + \frac{20}{x^6}$

Clearly $f''(\sqrt[6]{2}) = 2 + \frac{20}{2} > 0$

so f has a local min. at $x = \sqrt[6]{2}$

Because this was the only critical point, f must have an absolute min at $x = \sqrt[6]{2}$