Calculus 1 — Exam 7
MAT 250, Spring 2011 — D. Ivanšić

Name: Solution

Show all your work!

Find the following antiderivatives.

1. (4pts) $\int \cos\left(5x - \frac{\pi}{2}\right)\,dx =$ $\frac{1}{5}\sin\left(5x-\frac{\pi}{2}\right)+C$

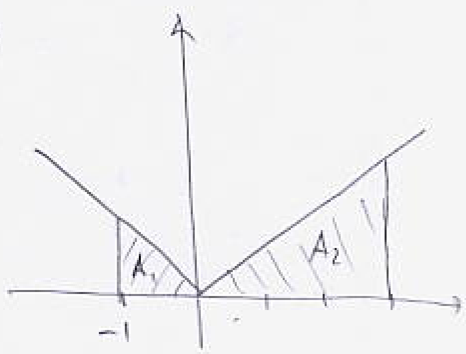
2. (7pts) $\int (x^3 - 4x)\sqrt{x}\,dx =$ $\int x^{\frac{7}{2}} - 4x^{\frac{3}{2}}\,dx = \frac{2}{9}x^{\frac{9}{2}} - 4\cdot\frac{2}{5}x^{\frac{5}{2}} = \frac{2}{9}x^{\frac{9}{2}} - \frac{8}{5}x^{\frac{5}{2}} + C$

$x^{\frac{1}{2}}$

3. (5pts) $\int \frac{1}{1+(3x)^2}\,dx =$ $\frac{1}{3}\arctan(3x)+C$

4. (16pts) Find $\int_{-1}^{3} |x|\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Fundamental Theorem of Calculus (you will have to break it up into two integrals).

a)



$\int_{-1}^{3} |x|\,dx = A_1 + A_2 = \frac{1}{2}1\cdot1 + \frac{1}{2}\cdot3\cdot3 = \frac{10}{2} = 5$

b) $\int_{-1}^{3} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{3} x\,dx$

$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{3} = -\frac{1}{2}(0-1) + \frac{1}{2}(9-0) = \frac{1}{2} + \frac{9}{2} = 5$

5. (6pts) Write in sigma notation.

$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} =$ $\sum_{i=1}^{5} (-1)^{i-1} \frac{i}{i+1}$

Use the substitution rule in the following integrals:

6. (9pts) $\int \frac{4x^3 + 14x}{\sqrt[5]{x^4 + 7x^2 + 9}}\, dx =$ $\left[\begin{array}{l} u = x^4 + 7x^2 + 9 \\ du = 4x^3 + 14x \end{array} \right] = \int \frac{du}{\sqrt[5]{u}} = \frac{5}{4} u^{\frac{4}{5}}$

$u^{-\frac{1}{5}}$

$= \frac{5}{4}\left(x^4 + 7x^2 + 9\right)^{\frac{4}{5}} + C$

7. (10pts) $\int_{\ln 3}^{\ln 5} \frac{e^x}{(7+e^x)^2}\, dx =$ $\left[\begin{array}{ll} u = 7 + e^x & x = \ln 5,\ u = 7 + e^{\ln 5} = 12 \\ du = e^x\, dx & x = \ln 3,\ u = 7 + e^{\ln 3} = 10 \end{array} \right]$

$= \int_{10}^{12} \frac{du}{u^2} = -\frac{1}{u}\Big|_{10}^{12} = -\left(\frac{1}{12} - \frac{1}{10}\right) = -\frac{5-6}{60} = \frac{1}{60}$

u^{-2}

8. (6pts) $\int_0^4 \frac{x-2}{\sin(x^2 - 4x + 7)}\, dx =$ $\left[\begin{array}{ll} u = x^2 - 4x + 7 & x = 4,\ u = 7 \\ du = (2x-4)\, dx & x = 0,\ u = 7 \\ \frac{du}{2} = (x-2)\, dx & \end{array} \right] = \int_7^7 \frac{1}{\sin u} \frac{du}{2}$

$= 0$ (because it is $\int_7^7$)

9. (8pts) A rocket shoots up vertically with velocity $v(t) = 5t^3 + 4t^2$ (in meters/second). Find its position function $s(t)$, if at time $t = 6$, the rocket is at altitude 1800m.

take antider.

$v(t) = 5t^3 + 4t^2$

$s(t) = 5\frac{t^4}{4} + 4\cdot\frac{t^3}{3} + C$

$1800 = s(6) = 5\cdot\frac{6^4}{4} + 4\cdot\frac{6^3}{3} + C$

$1800 = 5\cdot\frac{2^4\cdot 3^4}{4} + 4\cdot\frac{2^3\cdot 3^3}{3} + C$

$1800 = 20\cdot 81 + 32\cdot 9 + C$

$1620 + 288 = 1908$

$1800 = 1908 + C$

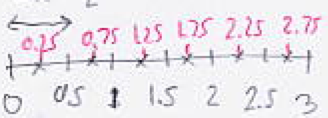
$C = -108$

$s(t) = \frac{5}{4}t^4 + 4\frac{t^3}{3} - 108$

10. (21pts) The function $f(x) = e^x$, $0 \le x \le 3$ is given.
a) Write down the expression that is used to compute M_6. Then compute M_6.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does M_6 represent?
c) Using the Fundamental Theorem of Calculus, evaluate $\int_0^3 e^x\,dx$. Is M_6 an overestimate or an underestimate of this integral?

$\Delta x = \frac{1}{2}$

a)



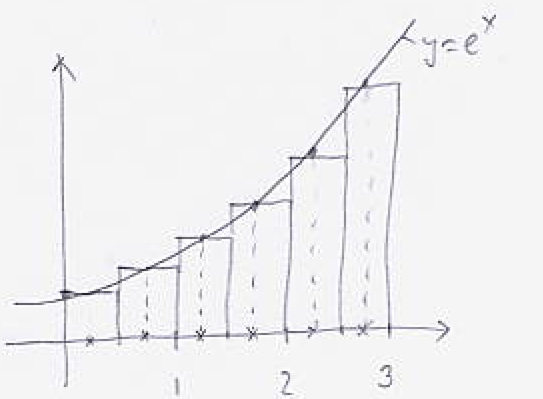
$M_6 = \frac{1}{2}\left(e^{0.25} + e^{0.75} + e^{1.25} + e^{1.75} + e^{2.25} + e^{2.75}\right)$

$= \frac{1}{2}\cdot 37.77.. = 18.888169$

c) $\int_0^3 e^x\,dx = e^x\Big|_0^3 = e^3 - 1$

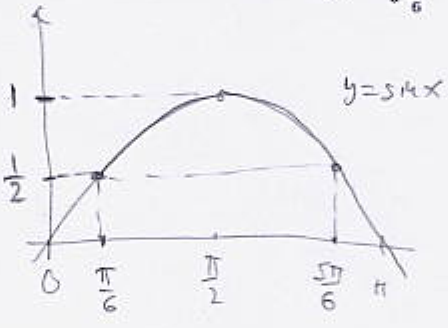
$= 19.085537$

$M_6 < \int_0^3 e^x\,dx$ so is an underestimate.



M_6 = area of the rectangles

11. (8pts) Show that $\frac{\pi}{3} \le \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x\, dx \le \frac{2\pi}{3}$ **without** evaluating the integral.



On $[\frac{\pi}{6}, \frac{5\pi}{6}]$ we have:

$\frac{1}{2} \le \sin x \le 1$ By comparison property:

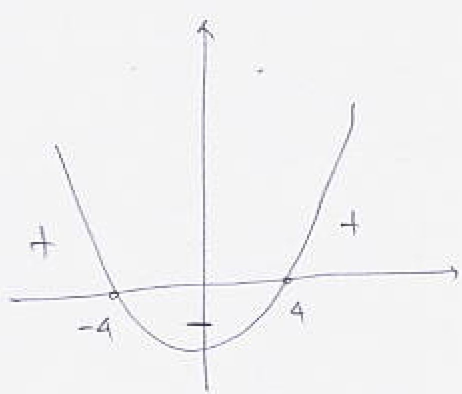
$$\frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) \le \int_{\pi/6}^{5\pi/6} \sin x\, dx \le 1 \cdot \left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$$

$$\frac{1}{2} \cdot \frac{2\pi}{3} \le \quad " \quad \le 1 \cdot \frac{2\pi}{3}$$

$$\frac{\pi}{3} \le \int_{\pi/6}^{5\pi/6} \sin x\, dx \le \frac{2\pi}{3}$$

Bonus. (10pts) The rate at which money flows in or out of a company's account is given by the formula $3t^2 - 48$ dollars/day, $0 \le t \le 30$, t in days. At time $t = 0$, there was \$800 in the account.
a) When is the company losing money from the account, and when is it gaining money?
b) By how much did the amount in the account increase or decrease from $t = 0$ to $t = 6$?
c) How much money is in the account when $t = 6$?

a) $A(t)$ = amount in account
$A'(t) = 3t^2 - 48$



Losing: on $(-4, 4)$
gaining on $(-\infty, -4) \cup (4, \infty)$

b) $\int_0^6 3t^2 - 48\, dt = (t^3 - 48t)\Big|_0^6$

$= 6^3 - 48 \cdot 6 = 6 \cdot (36 - 48) = -72 = \Delta A$

decreased by \$72.

c) $A(6) = A(0) + \Delta A = 800 + (-72)$

$= 728$