Calculus 1 — Exam 6
MAT 250, Spring 2011 — D. Ivanšić

Name: Solution

Show all your work!

Find the limits. Use L'Hopital's rule where appropriate.

1. (10pts) $\lim_{x\to\infty} \frac{\ln x}{\sqrt[10]{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{10}x^{-\frac{9}{10}}} = \lim_{x\to\infty} 10\frac{x^{-1}}{x^{-\frac{9}{10}}} = \lim_{x\to\infty} 10x^{-\frac{1}{10}} = \lim_{x\to\infty}\frac{10}{x^{1/10}} = \frac{10}{\infty} = 0$

$\sqrt[10]{x}$ grows faster since $\lim_{x\to\infty}\frac{\ln x}{\sqrt[10]{x}} = 0$

Which grows faster, $\ln x$ or $\sqrt[10]{x}$?

2. (8pts) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{e^x - 1}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

$1-1-0=0$, $\to 0$; $1-1\neq 0$, $\to 0$

3. (12pts) $\lim_{x\to 0+} \underbrace{(\sin x)^x}_{y} = \lim_{x\to 0+} e^{\ln y} = e^0 = 1$

$\ln y = \ln(\sin x)^x = x\ln(\sin x)$

$\lim_{x\to 0+} x\ln\sin x = \lim_{x\to 0+}\frac{\ln\sin x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0+}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}} = \lim_{x\to 0+} -\frac{x^2\cos x}{\sin x} \overset{L'H}{=} \lim_{x\to 0+} -\frac{2x\cos x + x^2(-\sin x)}{\cos x}$

$0\cdot(-\infty)$; $\to 0\cdot 1$; $\to 0$

$= -\frac{2\cdot 0\cdot 1 + 0\cdot 0}{1} = 0$

4. (8pts) $\lim_{x\to\infty}\sqrt{x^4+1} = \lim_{x\to\infty}\sqrt{x^4\left(1+\frac{1}{x^4}\right)} = \lim_{x\to\infty} \underbrace{x^2}_{\to\infty}\sqrt{1+\underbrace{\frac{1}{x^4}}_{\to 0}}$

$= \infty\cdot\sqrt{1+0} = \infty\cdot 1 = \infty$

5. (32pts) Let $f(x) = \dfrac{x^2}{x^2-4}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–d) to sketch the graph.

a) $y = \dfrac{x^2}{x^2-4}$

$$y' = \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2} = -\frac{8x}{\underbrace{(x^2-4)^2}_{>0}}$$

$y' = 0$ when $x = 0$

y' not def. when $x^2-4=0$

$x = \pm 2$

		-2		0		2	
y'	$+$	ND	$+$	0	$-$	ND	$-$
y	$\nearrow$	ND	$\nearrow$	loc. max	$\searrow$	ND	$\searrow$

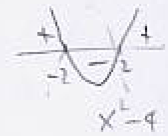
$$y'' = -8\,\frac{1\cdot(x^2-4)^2 - x\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}$$

$$= -8\,\frac{\cancel{(x^2-4)}(x^2-4-4x^2)}{(x^2-4)^{\cancel{4}\,3}}$$

$$= -8\,\frac{-3x^2-4}{(x^2-4)^3} = \frac{8\overbrace{(3x^2+4)}^{>0}}{(x^2-4)^3}$$

		-2		2	
y''	$+$	ND	$-$	ND	$+$
y	CU	ND	CD	ND	CU

Sign of y'' only depends on x^2-4



$y''=0$

$3x^2+4=0$

$x^2 = -\frac{4}{3}$

no sol.

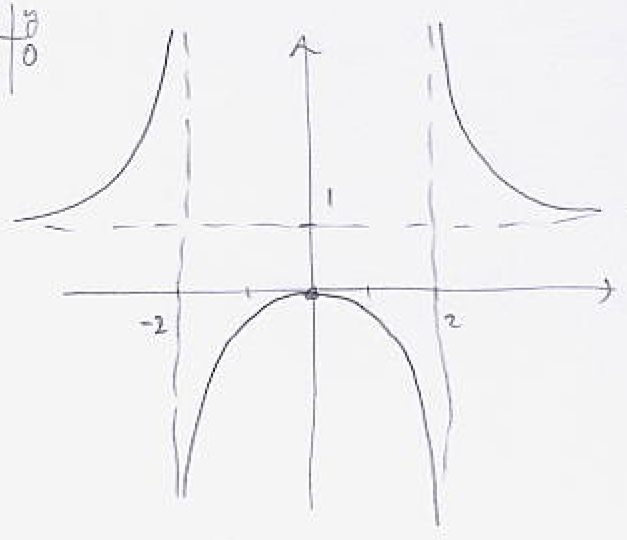
y'' not def.

$x = \pm 2$

y not def. for $x = \pm 2$

(has vertical asymptotes there)

x	y
0	0



c) $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2-4} = \lim\limits_{x\to\infty} \dfrac{\cancel{x^2}}{\cancel{x^2}(1-\underbrace{\frac{4}{x^2}}_{\to 0})} = \dfrac{1}{1-0} = 1$

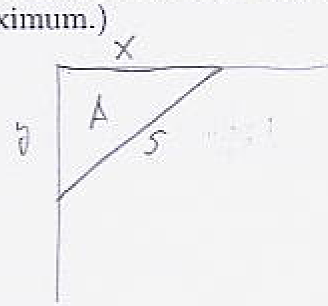
$x \to -\infty$

works same way

6. (8pts) Find the limit without using L'Hopital's rule.

$$\lim_{x\to\infty}\frac{5x^3+3x^2-x}{-x^2+7x-4} = \lim_{x\to\infty}\frac{x^3\left(5+\frac{3}{x}-\frac{1}{x^2}\right)}{x^2\left(-1+\frac{7}{x}-\frac{4}{x^2}\right)} = \left(\lim_{x\to\infty} x\right)\cdot\frac{5+0-0}{-1+0-0} = \infty\cdot(-5) = -\infty$$

7. (22pts) Jodie has a thick wooden pole 5 meters long that she will lay on the ground as a boundary for a triangular flower plot along the walls at a corner of her house (see picture).
a) Draw two more options for positioning the pole.
b) How should she position the pole so that the enclosed area is maximal? (Don't forget to check that it is a maximum.)



$x^2+y^2=5^2$

$y^2=25-x^2$

$y=\pm\sqrt{25-x^2}$

$(y\geq 0)$

$A=\frac{1}{2}xy=\frac{1}{2}x\sqrt{25-x^2}$

Maximizing A is same as maximizing A^2

$f(x)=A^2=\frac{1}{4}x^2(25-x^2)$

Job: maximize $\frac{1}{4}x^2(25-x^2)=\frac{1}{4}(25x^2-x^4)$ on $[0,5]$

$f'(x)=\frac{1}{4}(50x-4x^3)$

$50x-4x^3=0$

$x(50-4x^2)=0$

$x=0$ or $50-4x^2=0$

$x^2=\frac{50}{4}=\frac{25}{2}$

$x=\pm\frac{5}{\sqrt{2}}$

$(x\geq 0)$

x	$f(x)$
0	0
5	0
$\frac{5}{\sqrt{2}}$	$\frac{1}{4}\left(25\cdot\frac{25}{2}-\frac{5^4}{4}\right)=\frac{5^4}{16}=\frac{625}{16}>0$

Max. when $x=\frac{5}{\sqrt{2}}$

Bonus. (10pts) Show that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$. Note that this knowledge allows you to compute e approximately by putting a large x into the expression $\left(1+\frac{1}{x}\right)^x$ (although this is not a very efficient way to do it).

$$\lim_{x\to\infty} \underbrace{\left(1+\frac{a}{x}\right)^x}_{y} = \lim_{x\to\infty} e^{\ln y} = e^a$$

$$\ln y = \ln\left(1+\frac{a}{x}\right)^x = x\ln\left(1+\frac{a}{x}\right) \quad \ln(1+0)=0$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\underset{\to 0}{\frac{1}{x}}} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{1}{\left(1+\frac{a}{x}\right)}\cdot -\frac{a}{x^2}}{-\frac{1}{x^2}}\cdot\frac{-x^2}{-x^2}$$

$$= \lim_{x\to 0}\frac{a}{\left(1+\underset{\to 0}{\frac{a}{x}}\right)} = \frac{a}{1+0} = a$$

With a calculator

x	$\left(1+\frac{1}{x}\right)^x$
10	2.593742
100	2.704814
1000	2.716924
10000	2.718146

$e \approx 2.718282$ says calculator