Calculus 1 — Exam 5
MAT 250, Spring 2011 — D. Ivanšić

Name: Solution

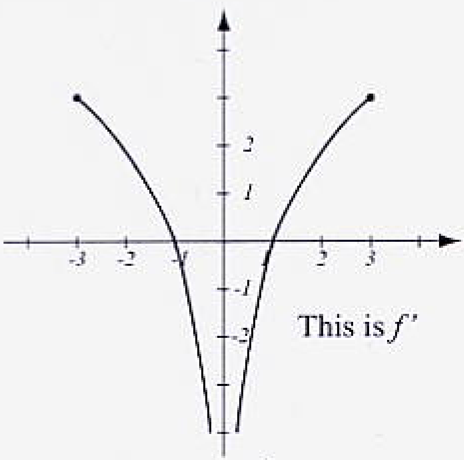
Show all your work!

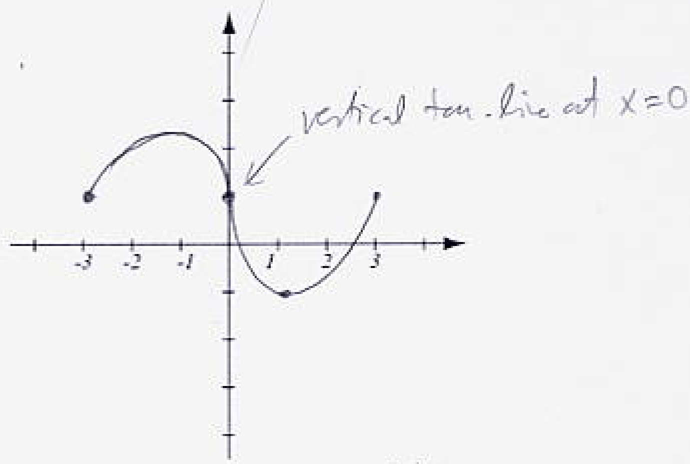
1. (18pts) Let the domain of f be $[-3, 3]$. The graph of its derivative f' is drawn below (note that f' is not defined at 0, the y-axis is its vertical asymptote). Use the graph to answer:

a) What are the intervals of increase and decrease of f? Where does f have a local minimum or maximum?

b) What are the intervals of concavity of f? Where does f have inflection points?

c) Use the information gathered in a) and b) to draw one possible graph of f at right. (Note that f *is* defined at 0.) with $f(0)=1$





a) f incr./decr. $\Leftrightarrow f' \gtrless 0$

f incr. on $(-3,-1) \cup (1,3)$

f decr. on $(-1,0) \cup (0,1)$

local max at $x=-1$

local min at $x=1$

b) f conc up / conc down $\Leftrightarrow f'$ incr./decr.

f conc up on $(0,3)$

f conc dn on $(-3,0)$

Inflection at $x=0$

2. (12pts) Use Rolle's Theorem to show that the equation $2x + \sin x = 0$ has at most one solution.

Suppose it has more than one, for example, a, b.

If $f(x) = 2x + \sin x$, this means $f(a) = 0 = f(b)$.

Since f is continuous and diff. on $\mathbb{R}$, Rolle's thm applies,

so there must be a $c \in (a,b)$ s.t. $f'(c) = 0$.

However: $f'(x) = 2 + \cos x$

$2 + \cos x = 0$

$\cos x = -2$

has no solution.

We arrive at a contradiction, so f has at most one solution.

3. (12pts) Verify the Mean Value Theorem for the function $f(x) = \ln x$ on the interval $[1, e^2]$. (Approximate e when necessary).

$e \approx 3$

$$\frac{f(e^2)-f(1)}{e^2-1} = \frac{\ln e^2 - \ln 1}{e^2-1} = \frac{2-0}{e^2-1} = \frac{2}{e^2-1}$$

$\ln x$ is cont. and diff. on $(0,\infty)$
so also on $[1,e^2]$

$f'(x) = \frac{1}{x}$ $\qquad \frac{1}{x} = \frac{2}{e^2-1}$ $\qquad x = \frac{e^2-1}{2} \in (1,e^2)$

$\approx \frac{3^2-1}{2} = 4 \in (1,9)$

4. (22pts) Let $f(x) = \sin^4 x + \cos^4 x$, where $x \in [0, \pi]$.
a) Find the critical points and the intervals of increase and decrease of f. Determine where f has a local maximum or minimum.
b) Using information found in a), draw a rough sketch of f.

a) $f'(x) = 4\sin^3 x\cos x + 4\cos^3 x(-\sin x)$

$= 4\sin x\cos x(\sin^2 x - \cos^2 x)$

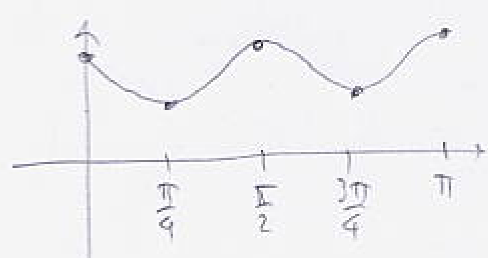
$y' = 0$ if

$\sin\theta = 0$ or $\cos\theta = 0$ or $\sin^2\theta = \cos^2\theta$

$\theta = 0, \pi$ $\qquad \theta = \frac{\pi}{2}$ $\qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	$\sin^4\theta + \cos^4\theta$
0	1
$\pi/4$	$(\frac{\sqrt2}{2})^4 + (\frac{\sqrt2}{2})^4 = \frac14 + \frac14 = \frac12$
$\pi/2$	1
$3\pi/4$	$(\frac{\sqrt2}{2})^4 + (-\frac{\sqrt2}{2})^4 = \frac12$
π	1

	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$		π
$\sin\theta$	0	+		+		+		+	0
$\cos\theta$		+		+	0	−		−	
$\sin^2\theta-\cos^2\theta$		−	0	+		+	0	−	
f'	0	−	0	+	0	−	0	+	0
f		↘		↗		↘		↗	



5. (16pts) Let $f(x) = \dfrac{x+3}{x^2+7}$. Find the absolute minimum and maximum values of f on the interval $[-1, 2]$.

$$f'(x) = \frac{1\cdot(x^2+7) - (x+3)\cdot 2x}{(x^2+7)^2} = \frac{x^2+7-(2x^2+6x)}{(x^2+7)^2} = \frac{-x^2-6x+7}{(x^2+7)^2}$$

$f'(x) = 0$ when $-x^2-6x+7=0$

$$x^2+6x-7=0$$
$$(x+7)(x-1)=0$$
$$x = -7, 1$$

not in interval

x	$\frac{x+3}{x^2+7}$	
1	$\frac{4}{8} = \frac{1}{2}$	abs max
-1	$\frac{2}{8} = \frac{1}{4}$	abs min
2	$\frac{5}{11}$	

6. (20pts) Let $f(x) = x^2e^x$.

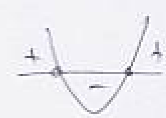
a) Find the intervals of concavity and points of inflection for f.

b) Find the critical points and use a) to determine which are local minima or maxima.

a) $f'(x) = 2xe^x + x^2e^x$
$$= (x^2+2x)e^x$$

$f''(x) = (2x+2)e^x + (x^2+2x)e^x$
$$= (x^2+4x+2)e^x$$

		$-2-\sqrt{2}$		$-2+\sqrt{2}$	
f''	$+$	0	$-$	0	$+$
f	CU	IP	CD	IP	CU



2nd order crit. pts:

$$(x^2+4x+2)\underbrace{e^x}_{>0} = 0$$

$$x^2+4x+2=0$$

$$x = \frac{-4 \pm \sqrt{16-4\cdot2\cdot1}}{2} = \frac{-4\pm\sqrt{8}}{2}$$

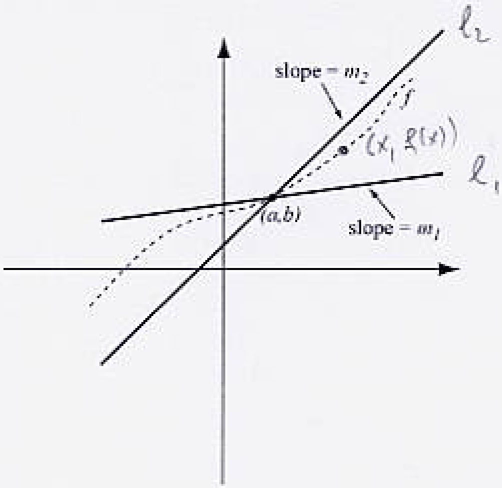
$$= \frac{-4\pm 2\sqrt{2}}{2} = -2\pm\sqrt{2}$$

b) Critical pts: $(x^2+2x)\underbrace{e^x}_{>0} = 0$

$$x^2+2x=0$$
$$x(x+2)=0$$
$$x=0,-2$$

x	$f''(x) = (x^2+4x+2)\cdot e^x$	
0	$2\cdot(+) = +$	loc. min
-2	$-2(+) = -$	loc. max.

Bonus. (10pts) Let f be differentiable for all real numbers x so that $m_1 \leq f'(x) \leq m_2$ and $f(a) = b$. Use the Mean Value Theorem to show that the graph of f must be between the two lines with slopes m_1 and m_2, shown in the picture.



Equations of lines:

l_1: $y - b = m_1(x-a)$

$y = b + m_1(x-a)$

l_2 $y - b = m_2(x-a)$

$y = b + m_2(x-a)$

By MVT, for any x, there is a $c \in (a,x)$

s.t. $f'(c) = \dfrac{f(x)-f(a)}{x-a}$

Since $m_1 \leq f'(c) \leq m_2$ this means that

$$m_1 \leq \frac{f(x)-f(a)}{x-a} \leq m_2 \quad \Big| \cdot x-a$$

$$m_1(x-a) \leq f(x) - f(a) < m_2(x-a)$$

$$\downarrow$$

$$m_1(x-a) \leq f(x) - b < m_2(x-a)$$

$$\underbrace{b + m_1(x-a)}_{\text{line } l_1} < f(x) \leq \underbrace{b + m_2(x-a)}_{\text{line } l_2}$$