Calculus 1 — Exam 4
MAT 250, Spring 2011 — D. Ivanšić

Name: Solution

Show all your work!

Differentiate and simplify where appropriate:

1. (4pts) $\frac{d}{dx}\sqrt{x^3-4x^2+1} = \frac{1}{2\sqrt{x^3-4x^2+1}}(3x^2-8x) = \frac{3x^2-8x}{2\sqrt{x^3-4x^2+1}}$

2. (6pts) $\frac{d}{dx}e^{3x}\sin 5x = e^{3x}\cdot 3\sin 5x + e^{3x}\cdot\cos 5x\cdot 5 = e^{3x}(3\sin 5x + 5\cos 5x)$

3. (7pts) $\frac{d}{du}e^{\frac{1}{u^2+5u}} = \frac{d}{du}e^{(u^2+5u)^{-1}} = e^{(u^2+5u)^{-1}}(-1)(u^2+5u)^{-2}(2u+5) = -\frac{(2u+5)e^{\frac{1}{u^2+5u}}}{(u^2+5u)^2}$

4. (8pts) $\frac{d}{dx}\ln\frac{x^2+1}{(x+4)^2} = \frac{d}{dx}\left(\ln(x^2+1) - 2\ln(x+4)\right) = \frac{1}{x^2+1}\cdot 2x - 2\cdot\frac{1}{x+4}$

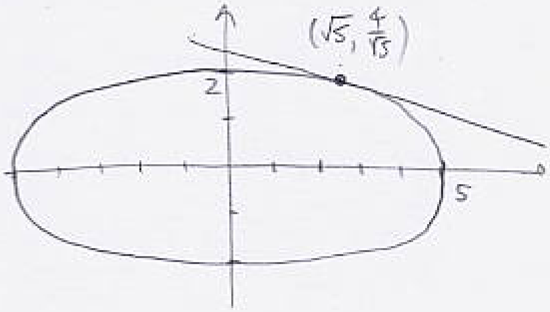
$= \frac{2x(x+4)-2(x^2+1)}{(x^2+1)(x+4)} = \frac{8x-2}{(x^2+1)(x+4)}$

5. (6pts) $\frac{d}{dy}7^{7^y} = \ln 7\cdot 7^{7^y}\cdot(7^y)' = \ln 7\cdot 7^{7^y}\ln 7\cdot 7^y = (\ln 7)^2\cdot 7^{7^y}\cdot 7^y$

6. (9pts) $\frac{d}{dx}\frac{\arccos x}{\sqrt{1-x^2}} = \frac{-\frac{1}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2} - \arccos x\cdot\left(\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\right)}{(\sqrt{1-x^2})^2} = \frac{-1+\frac{x\arccos x}{\sqrt{1-x^2}}}{1-x^2}\cdot\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$

$= \frac{x\arccos x - \sqrt{1-x^2}}{(1-x^2)^{3/2}}$

7. (12pts) Use implicit differentiation to find the equation of the tangent line to the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at the point $(\sqrt{5}, \frac{4}{\sqrt{5}})$. Draw the picture of the ellipse and the tangent line.



$$\frac{x^2}{25} + \frac{y^2}{4} = 1 \quad \Big| \frac{d}{dx}$$

$$\frac{1}{25} \cdot 2x + \frac{1}{4} \cdot 2yy' = 0$$

$$\frac{yy'}{2} = -\frac{2}{25}x \quad \Big| \cdot \frac{2}{y}$$

$$y' = -\frac{4x}{25y}$$

$$y'\Big|_{\substack{x=\sqrt{5} \\ y=\frac{4}{\sqrt{5}}}} = -\frac{4 \cdot \sqrt{5}}{25 \cdot \frac{4}{\sqrt{5}}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{20}{100} = -\frac{1}{5}$$

Equation of tan. line:

$$y - \frac{4}{\sqrt{5}} = -\frac{1}{5}(x - \sqrt{5})$$

$$y = -\frac{1}{5}x + \frac{\sqrt{5}}{5} + \frac{4}{\sqrt{5}}$$

$$y = -\frac{1}{5}x + \frac{\sqrt{5}}{5} + \frac{4\sqrt{5}}{5}$$

$$y = -\frac{1}{5}x + \frac{5\sqrt{5}}{5} = -\frac{1}{5}x + \sqrt{5}$$

8. (10pts) Use implicit differentiation to find y'.

$$x^4 + y^4 = \frac{x}{y} \quad \Big| \frac{d}{dx}$$

$$4x^3 + 4y^3y' = \frac{1 \cdot y - x \cdot y'}{y^2} \quad \Big| \cdot y^2$$

$$4x^3y^2 + 4y^5y' = y - xy'$$

$$4y^5y' + xy' = y - 4x^3y^2$$

$$y'(4y^5 + x) = y - 4x^3y^2$$

$$y' = \frac{y - 4x^3y^2}{x + 4y^5}$$

9. (10pts) Use logarithmic differentiation to find the derivative of $y = (\arctan x)^x$.

$$y = (\arctan x)^x \quad \Big| \ln$$

$$\ln y = \ln(\arctan x)^x$$

$$\ln y = x \ln(\arctan x) \quad \Big| \frac{d}{dx}$$

$$\frac{1}{y} \cdot y' = 1 \cdot \ln(\arctan x) + x \cdot \frac{1}{\arctan x} \cdot \frac{1}{x^2+1} \quad \Big| \cdot y$$

$$y' = \left(\ln(\arctan x) + \frac{x}{(x^2+1)\arctan x}\right) \cdot (\arctan x)^x$$

10. (12pts) Let $f(x) = x^2 - 4x$, $x \geq 2$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find $g'(12)$.

$$x \underset{g}{\overset{f}{\rightleftarrows}} 12$$

$$g'(12) = \frac{1}{f'(g(12))} = \frac{1}{f'(6)} = \frac{1}{2 \cdot 6 - 4} = \frac{1}{8}$$

$g(12)$ is solution to:

$$f(x) = 12$$

$$f'(x) = 2x - 4$$

$$x^2 - 4x = 12$$

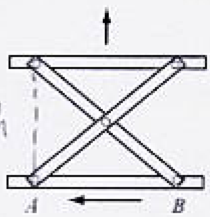
$$x^2 - 4x - 12 = 0$$

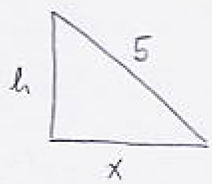
$$(x-6)(x+2) = 0$$

$$x = 6, \underbrace{-2}_{\text{not} \geq 2}$$

$$x = 6$$

11. (16pts) A workers' platform raises as point B is pulled by a chain towards the stationary point A (circles denote rotating joints). If the diagonal beams are 5 meters long, and B is pulled towards A at rate 2 meters per minute, at what rate is the top of the platform rising when B is 3 meters away from A? What are the units? *(Hint: use a triangle.)*





Need: $\frac{dh}{dt}$ when $x=3$

Know: $\frac{dx}{dt} = -2$ m/min

$x^2+h^2 = 5^2 \quad \Big| \frac{d}{dt}$

When $x=3$, $3^2+h^2=5^2$

$9+h^2=25$

$h^2=16$

$h=4$

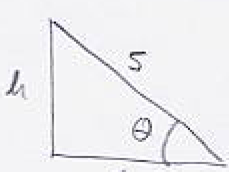
$2xx' + 2hh' = 0$

$2hh' = -2xx'$

$h' = -\frac{2xx'}{2h} = -\frac{xx'}{h}$

$h' = -\frac{3\cdot(-2)}{4} = \frac{3}{2}$ m/min

Bonus. (10pts) In the previous problem let θ be the angle between either of the diagonal beams and the horizontal. Find how fast θ is increasing at the moment described above.

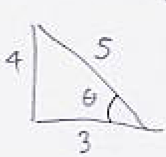


$\cos\theta = \frac{x}{5} \quad \Big| \frac{d}{dt}$

$-\sin\theta \cdot \theta' = \frac{x'}{5}$

$\theta' = -\frac{x'}{5\sin\theta}$

When $x=3$, $h=4$ (above)



$\sin\theta = \frac{4}{5}$

$\theta' = -\frac{-2}{5\cdot\frac{4}{5}} = \frac{2}{4} = \frac{1}{2}$ rad/min