Calculus 1 — Exam 3
MAT 250, Spring 2011 — D. Ivanšić

Name: Solution

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left((x^4-4x)\sin x\right) =$ $(4x^3-4)\sin x + (x^4-4x)\cos x$

2. (8pts) $\dfrac{d}{dx}\dfrac{5x+3}{x^2-7x+4} =$ $\dfrac{5\cdot(x^2-7x+4)-(5x+3)(2x-7)}{(x^2-7x+4)^2} = \dfrac{5x^2-35x+20-(10x^2-29x-21)}{(x^2-7x+4)^2}$

$= \dfrac{-5x^2-6x+41}{(x^2-7x+4)^2}$

3. (8pts) $\dfrac{d}{dz}\dfrac{z+\sqrt{z}}{z-\sqrt{z}} =$ $\dfrac{\left(1+\frac{1}{2\sqrt{z}}\right)(z-\sqrt{z})-(z+\sqrt{z})\left(1-\frac{1}{2\sqrt{z}}\right)}{(z-\sqrt{z})^2}$

$\dfrac{z}{\sqrt{z}} = \sqrt{z}$

$= \dfrac{\left(\cancel{z}+\frac{z}{2\sqrt{z}}-\sqrt{z}-\cancel{\frac{1}{2}}\right)-\left(\cancel{z}+\sqrt{z}-\frac{z}{2\sqrt{z}}-\cancel{\frac{1}{2}}\right)}{(z-\sqrt{z})^2} = \dfrac{\frac{1}{2}\sqrt{z}-\sqrt{z}-\sqrt{z}+\frac{1}{2}\sqrt{z}}{(z-\sqrt{z})^2} = -\dfrac{\sqrt{z}}{(z-\sqrt{z})^2}$

4. (8pts) $\dfrac{d}{d\theta}\dfrac{\cos\theta-1}{\sin\theta} =$ $\dfrac{-\sin\theta\sin\theta-(\cos\theta-1)\cos\theta}{\sin^2\theta} = \dfrac{-\sin^2\theta-\cos^2\theta+\cos\theta}{\sin^2\theta}$

$= \dfrac{\cos\theta-1}{\sin^2\theta} = \dfrac{\cancel{\cos\theta-1}^{\,-1}}{(\cancel{1-\cos\theta})(1+\cos\theta)} = -\dfrac{1}{1+\cos\theta}$

$1-\cos^2\theta$

5. (8pts) Let $f(3)=2$, $f'(3)=-1$, $g(3)=4$ and $g'(3)=-2$, and let $h(x)=\dfrac{xf(x)}{g(x)}$.

a) Find the general expression for $h'(x)$.

b) Find $h'(3)$.

a) $h'(x) = \dfrac{(xf(x))'g(x)-xf(x)g'(x)}{(g(x))^2} = \dfrac{(f(x)+xf'(x))g(x)-xf(x)g'(x)}{(g(x))^2}$

$= \dfrac{f(x)g(x)+xf'(x)g(x)-xf(x)g'(x)}{(g(x))^2}$

b) $h'(3) = \dfrac{2\cdot 4+3\cdot(-1)\cdot 4-3\cdot 2\cdot(-2)}{4^2}$

$= \dfrac{8-12+12}{16} = \dfrac{1}{2}$

6. (10pts) Find the equation of the tangent line to the curve $y = \tan^2 x$ at the point $x = \frac{\pi}{4}$.

$y' = (\tan x \cdot \tan x)'$

$= \sec^2 x \tan x + \tan x \cdot \sec^2 x$

$= 2 \tan x \sec^2 x$

$y(\frac{\pi}{4}) = 1^2 = 1$

$y'(\frac{\pi}{4}) = 2 \tan\frac{\pi}{4} \cdot \frac{1}{\cos^2\frac{\pi}{4}} = 2 \cdot 1 \cdot \frac{1}{(\frac{1}{\sqrt{2}})^2} = 4$

$y - 1 = 4(x - \frac{\pi}{4})$

$y = 4x - \pi + 1$ eq. of tan. line

7. (16pts) A pomegranate is thrown upwards so that at height 30m it has upward velocity 10m/s.

a) Write the formula for the position of the pomegranate at time t (you may assume $g \approx 10$ and take $t = 0$ to be the time of the above observation).

b) When does the pomegranate reach height 15m on the way down? On the way up?

c) Write the formula for the velocity of the pomegranate at time t.

d) What are the velocities of the pomegranate at the times from b)?

a) $s(t) = 30 + 10t - 5t^2$

b) $s(t) = 15$

$30 + 10t - 5t^2 = 15$

$5t^2 - 10t - 15 = 0 \quad | \div 5$

$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$t = -1, 3$

$\uparrow$ on way up $\quad \uparrow$ on way down

c) $v(t) = s'(t) = 10 - 10t$

d) $v(-1) = 10 - 10 \cdot (-1) = 20$ m/s

$v(3) = 10 - 10 \cdot 3 = -20$ m/s

8. (14pts) The volume in cm^3 of a cantaloup is given by the formula $V = \frac{1}{10}A^{\frac{3}{2}}$, where A is its surface area in cm^2.
a) Find the volume of a cantaloup whose surface area is 900cm^2.
b) Find the ROC of volume with respect to surface area when $A = 900$ (units?).
c) Use b) to estimate the change in volume if surface area decreases by 50cm^2.
d) Use c) to estimate the volume of a cantaloup with surface area 850cm^2 and compare to the actual value of 2478.1546cm^3.

a) $V(900) = \frac{1}{10} 900^{\frac{3}{2}} = \frac{1}{10}\left(\sqrt{900}\right)^3 = \frac{1}{10} 30^3 = \frac{1}{10} \cdot 27000 = 2700$

b) $\frac{dV}{dA} = \frac{1}{10} \frac{3}{2} A^{\frac{1}{2}} = \frac{3}{20}\sqrt{A}$

$\frac{dV}{dA}(900) = \frac{3}{20} \cdot \sqrt{900} = \frac{3}{20} \cdot 30 = 4.5 \quad \text{cm}^3/\text{cm}^2$

c) $\Delta V \approx V'(900) \cdot \Delta A = 4.5 \cdot (-50) = -225 \text{ cm}^3$

d) $V(850) \approx V(900) + \Delta V = 2700 - 225 = 2475$, fairly close to actual value

9. (12pts) Let $f(x) = x^{-1}$.
a) Find the first four derivatives of f.
b) Find the general formula for $f^{(n)}(x)$.

a) $f'(x) = (-1)x^{-2}$

$f''(x) = (-1)(-2)x^{-3}$

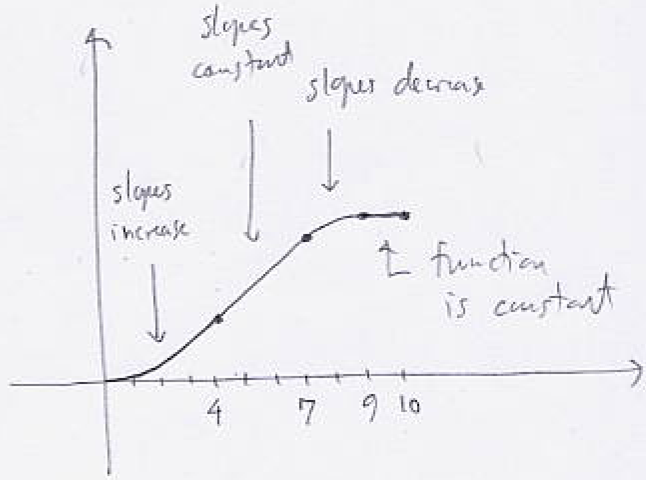
$f'''(x) = (-1)(-2)(-3)x^{-4}$

$f^{(4)}(x) = (-1)(-2)(-3)(-4)x^{-5}$

$f^{(n)}(x) = (-1)(-2)(-3)\ldots(-n)\, x^{-(n+1)}$

$= (-1)^n n!\, x^{-(n+1)}$

10. (10pts) An automobile's position is tracked for 10 seconds. Draw the graph of its position function if we know the following:
— it is always moving forward
— it accelerates on interval $(0, 4)$
— it moves at a steady velocity on interval $(4, 7)$
— it decelerates on interval $(7, 9)$
— it is at rest on interval $(9, 10)$.



Bonus. (10pts) Let $f(x) = e^x \sin x$.
a) Find the first four derivatives of f.
b) Find the pattern for $f^{(n)}(x)$. (You may need to describe it in words.)
c) What is $f^{(35)}(x)$?

a) $y = e^x \sin x$

$y' = e^x \sin x + e^x \cos x$

$= e^x(\sin x + \cos x)$

$y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= 2e^x \cos x$

$y''' = 2\left(e^x \cos x + e^x(-\sin x)\right)$

$= 2e^x(\cos x - \sin x)$

$y^{(4)} = 2\left(e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)\right)$

$= 2e^x(-2\sin x) = -4\underbrace{e^x \sin x}$

same function as at beginning

b) Derivatives come in cycles of functions:

$m \cdot e^x(\sin x + \cos x)$

$m \cdot 2e^x \cos x$

$m \cdot 2e^x(\cos x - \sin x)$

$m \cdot 4e^x \sin x$

where m is a power of (-4)

For k-th cycle, $m = (-4)^{k-1}$

c) $y^{(35)}$: $35 = 32 + 3$

8 cycle

$y^{(35)}$ is third in ninth cycle,

so it is $(-4)^7 \cdot 2e^x(\cos x - \sin x)$

$= -2 \cdot 4^7 e^x(\cos x - \sin x)$