Calculus 1 — Exam 2 **MAT 250, Spring 2011 — D. Ivanšić**

Name: Solution

Show all your work!

1. (17pts) Differentiate and simplify where appropriate:

$$\frac{d}{dx}\left(4x^3 - \frac{3}{x^6} + \sqrt[3]{x^{13}} + 4^5\right) = \frac{d}{dx}\left(4x^3 - 3x^{-6} + x^{\frac{13}{3}} + \underbrace{4^5}_{\text{constant}}\right)$$

$$= 4\cdot 3x^2 - 3\cdot(-6)x^{-7} + \frac{13}{3}x^{\frac{13}{3}-1}$$

$$= 12x^2 + 18x^{-7} + \frac{13}{3}x^{\frac{10}{3}}$$

$$\frac{d}{dx}\left((\sqrt{x}+7)(3\sqrt[4]{x} - x^2)\right) = \frac{d}{dx}\left((x^{\frac{1}{2}}+7)(3x^{\frac{1}{4}} - x^2)\right) = \frac{d}{dx}\left(3x^{\frac{3}{4}} + 21x^{\frac{1}{4}} - x^{\frac{5}{2}} - 7x^2\right)$$

$$= 3\cdot\frac{3}{4}x^{\frac{3}{4}-1} + 21\cdot\frac{1}{4}x^{\frac{1}{4}-1} - \frac{5}{2}x^{\frac{5}{2}-1} - 7\cdot 2x$$

$$= \frac{9}{4}x^{-\frac{1}{4}} + \frac{21}{4}x^{-\frac{3}{4}} - \frac{5}{2}x^{\frac{3}{2}} - 14x$$

$$\frac{d}{dy}\left(4e^y + \underbrace{3e^4}_{\text{constant}}\right) = 4e^y$$

2. (10pts) Use the Intermediate Value Theorem to show that the equation $\cos x = x - 1$ has at least one solution. Write a nice sentence that shows how you are using the IVT.

$\cos x = x - 1$

$\cos x - x = -1$

Let $f(x) = \cos x - x$, which is continuous on $\mathbb{R}$.

$f(0) = \cos 0 - 0 = 1$

$f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} \approx -1.5$

Since $f(\frac{\pi}{2}) < -1 < f(0)$, by the IVT there is a $c \in (0, \frac{\pi}{2})$ so that $f(c) = -1$

3. (22pts) Find the following limits algebraically.

$$\lim_{x\to -4} \frac{x^2-3x-28}{x^2-16} = \lim_{x\to -4} \frac{(x-7)\cancel{(x+4)}}{(x-4)\cancel{(x+4)}} = \lim_{x\to -4} \frac{x-7}{x-4} = \frac{-11}{-8} = \frac{11}{8}$$

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+3}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

$$\lim_{x\to 0} \frac{\sin(7x)}{\sin(5x)} = \lim_{x\to 0} \frac{\sin 7x}{7x}\cdot 7x \cdot \frac{5x}{\sin 5x}\,\frac{1}{5x} = \underbrace{\lim_{x\to 0}\frac{\sin 7x}{7x}}_{=1}\cdot\underbrace{\lim_{x\to 0}\frac{5x}{\sin 5x}}_{=1}\cdot\frac{7}{5}$$

$$= \frac{7}{5}$$

4. (10pts) Find $\lim_{x\to 0} x^2\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)$. Use the theorem that rhymes with what you pay at the bursar's office, other than tuition.

$-1 \le \sin\frac{1}{x} \le 1$

$-1 \le \cos\frac{1}{x} \le 1$

$-2 \le \sin\frac{1}{x}+\cos\frac{1}{x} \le 2 \quad |\cdot x^2$

$-2x^2 \le x^2\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right) \le 2x^2$

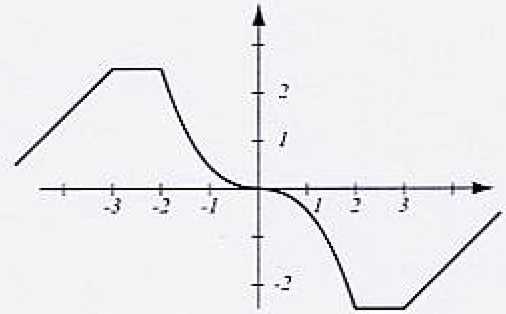
$$\left.\begin{array}{l}\lim_{x\to 0} -2x^2 = 0\\ \lim_{x\to 0} 2x^2 = 0\end{array}\right\}\ \text{are equal}$$

By the squeeze theorem, we conclude that

$$\lim_{x\to 0} x^2\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right) = 0.$$

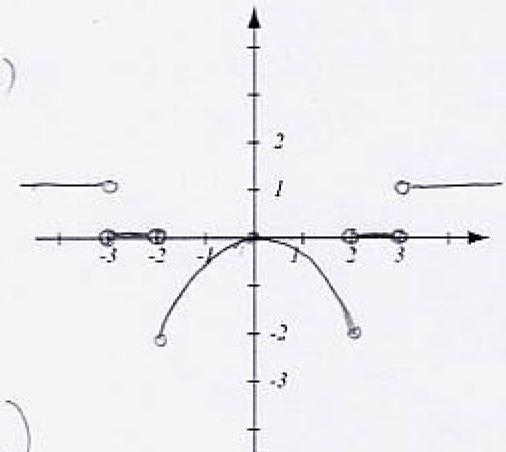
5. (15pts) The graph of the function $f(x)$ is shown at right.

a) Find the points where $f'(a)$ does not exist.
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.
c) Is $f(x)$ odd or even? How about $f'(x)$?



a) $x = -3, -2, 2, 3$ (sharp points)

b)



c) f is odd

f' is even

(This is, incidentally, always true: if f is odd, f' is even, and vice versa.)

6. (16pts) Let $f(x) = \dfrac{1}{x^2}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(2, \frac{1}{4})$.

a) $$f'(a) = \lim_{x\to a} \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x-a} = \lim_{x\to a} \frac{\frac{a^2-x^2}{x^2a^2}}{x-a} = \lim_{x\to a} \frac{a^2-x^2}{x^2a^2}\cdot\frac{1}{x-a}$$

$$= \lim_{x\to a} \frac{\overbrace{(a-x)}^{=-\cancel{(x-a)}}(a+x)}{x^2a^2\cancel{(x-a)}} = \lim_{x\to a} \frac{-(a+x)}{x^2a^2} = -\frac{2a}{a^4} = -\frac{2}{a^3}$$

b) $f(x) = x^{-2}$

$f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$, agrees with a)

c) $f'(2) = -\dfrac{2}{2^3} = -\dfrac{1}{4}$

$y - \frac{1}{4} = -\frac{1}{4}(x-2)$ tangent line

$y = -\frac{1}{4}x + \frac{1}{2} + \frac{1}{4}$

$y = -\frac{1}{4}x + \frac{3}{4}$

7. (10pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Use the infomation above to find the limit.

$$\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

a) $f(x) = \sqrt[3]{x}$ $\quad (\sqrt[3]{8}=2)$

$a=8$

b) The limit is $f'(8)$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, so $f'(8) = \frac{1}{3}8^{-\frac{2}{3}} = \frac{1}{3}\frac{1}{8^{\frac{2}{3}}} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

Bonus. (10pts) Use the limit definition of the derivative to find the derivative of the function $f(x) = \dfrac{x^2}{x+3}$.

$$\lim_{x\to a} \frac{\frac{x^2}{x+3}-\frac{a^2}{a+3}}{x-a} = \lim_{x\to a} \frac{\frac{x^2(a+3)-a^2(x+3)}{(x+3)(a+3)}}{x-a} = \lim_{x\to a} \frac{x^2a+3x^2-a^2x-3a^2}{(x+3)(a+3)}\cdot\frac{1}{x-a}$$

$$= \lim_{x\to a} \frac{x^2a-a^2x+3x^2-3a^2}{(x+3)(a+3)(x-a)} = \lim_{x\to a} \frac{ax(x-a)+3(x-a)(x+a)}{(x+3)(a+3)(x-a)}$$

$$= \lim_{x\to a} \frac{(ax+3)\cancel{(x-a)}}{(x+3)(a+3)\cancel{(x-a)}} = \frac{a\cdot a+3}{(a+3)(a+3)} = \frac{a^2+3}{(a+3)^2}$$

Thus, $f'(x) = \dfrac{x^2+3}{(\cancel{a}x+3)^2}$