Mathematical Reasoning — Exam 1
MAT 312, Fall 2011 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (17pts) Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is a predicate, find its truth set.

a) (universal set=$\mathbf{R}$) $x^2 - 5x + 6 = 0$

b) If an integer is divisible by 4, then it is divisible by 6.

c) If pigs fly, then the sun sets in the east.

d) There exists a real number x such that $x^2 - 7 = 0$.

e) For every $x \in \mathbf{R}$, if $x > 1$, then $x^4 > x^2$.

2. (8pts) Negate the following statements.

a) Leia does not answer the questions and she suffers the consequences.

b) If Luke goes to the Dagobah system, he gets stuck in the swamp or masters the force.

3. (8pts) Use a truth table to prove the equivalence $P \iff Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$. (Use however many columns you need.)

P	Q								
T	T								
T	F								
F	T								
F	F								

4. (12pts) Use previously proven logical equivalences to prove the equivalence $P \implies (Q \implies R) \equiv (P \wedge Q) \implies R$. Do not use a truth table.

5. (4pts) Write the converse and contrapositive of the statement: If $x > 0$, then $3^x > 1$.

Converse:

Contrapositive:

6. (4pts) Use the roster method to write the set $\{x \in \mathbf{Z} \mid x^2 + 4 < 10\}$.

7. (6pts) Use set builder notation to write the set $\{\ldots, -5, -1, 3, 7, 11, \ldots\}$

8. (14pts) For each of the following statements, do the following:
a) Write the statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

1) There exist real numbers x and y such that $\sin x + \sin y = 3$.

2) There exists a positive real number y such that for every real number x, if $x^2 + y < 7$, then $x^2 + y^2 > 25$.

9. (12pts) Let $\mathbf{R}$ be the universal set. The following is a predicate in x:

$$(\exists y \in \mathbf{R})(x^2 - y^2 = 16)$$

a) If $x = 3$, is the statement true?
b) If $x = 7$, is the statement true?
c) Find the truth set (the x's) of the above statement. Write it using interval notation.

10. (15pts) An integer n is called a type-0, type-1 or type-2 integer if it can be written in the form $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-2 integer, then $m^2 + mn + n^2$ is a type-1 integer. Start with a know-show table if you find it helpful.

Bonus. (10pts) Determine whether the statements 1 and 2 in problem 8 are true and justify.

Mathematical Reasoning — Exam 2 MAT 312, Fall 2011 — D. Ivanšić

Name: ____________________

Show all your work!

1. (12pts) Let U be the set of real numbers. Consider the intervals $A = [-3, 2]$, $B = [-1, 0]$, $C = (0, \infty)$ and write the following subsets in interval notation (draw the real line if it helps):

$A \cap C$ $\qquad$ $A \cup C$ $\qquad$ $A - B$ $\qquad$ $B - C$ $\qquad$ A^c $\qquad$ $A \cap B \cap C$

2. (18pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets.
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A - B) - C$ $\qquad$ $A - (B - C)$ $\qquad$ $A - (B \cap C)$ $\qquad$ $A - (B \cup C)$

3. (14pts) Let $A = \{x \in \mathbf{Z} \mid x \equiv 2 \pmod 3\}$ and $B = \{x \in \mathbf{Z} \mid x \equiv 5 \pmod 6\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

4. (10pts) Prove: for every real number x, if x is irrational, then $\frac{1}{x}$ is irrational.

5. (14pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$. (Note: one direction can be done simply by set algebra.)

6. (18pts) Prove the following:
a) For every integer a, if a^3 is divisible by 3, then a is divisible by 3.
b) $\sqrt[3]{9}$ is an irrational number. (Use statement a)).

7. (14pts) Prove that for every real number $a \geq 0$, $a + \dfrac{1}{a} \geq 2$.

Bonus. (10pts) Prove that 131,739,418 is not a square of any integer.

Mathematical Reasoning — Exam 3
MAT 312, Fall 2011 — D. Ivanšić

Name:______________________________

Show all your work!

1. (6pts) Draw arrow diagrams between two sets that illustrate

a) a surjection b) an injection that is not a surjection c) something that is not a function

2. (14pts) Let $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, and let $f : \mathbf{Z}_6 \to \mathbf{Z}_6$, $f(x) = 2x + 1 \pmod 6$.
a) Write the table of function values.
b) What is the set of preimages of 3?
c) What is the set of preimages of 4?
d) Is f injective? Justify.
e) Is f surjective? Justify.

3. (16pts) Use induction to prove: for every $n \in \mathbf{N}$ and every $x \in \mathbf{R}$, $x \neq 1$,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

4. (18pts) Let $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be given by $g(m, n) = (2m + n, 5m + 3n)$.
a) Evaluate $g(3, -4)$ and $g(-1, -3)$.
b) Find the set of preimages of $(-1, 6)$.
c) Show that g is surjective.

5. (18pts) Let $f(x) = \dfrac{4}{x^2+1}$. We assume the codomain is $\mathbf{R}$.

a) What subset of real numbers is the natural domain for this function?

b) Is this function injective? Justify.

c) What is the range of this function? Justify your answer.

6. (8pts) Let $h : \mathbf{R} - \{\frac{1}{3}\} \to \mathbf{R}$ be given by $h(x) = \dfrac{2x}{3x-1}$. Show that h is injective.

7. (20pts) Use induction to prove: when dividing by 3 the sum of squares of three consecutive natural numbers one always gets remainder 2. (For example, dividing $7^2+8^2+9^2$ by 3 gives remainder 2.)

Bonus. (10pts) Construct a bijection $\mathbf{N} \to \mathbf{Z}$. Use an arrow diagram to get an idea.

Mathematical Reasoning — Final Exam
MAT 312, Fall 2011 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) Are the following statements true or false? Justify.

a) If an integer is divisible by 6, then it is divisible by 3.

b) If roses are blue, then violets are red.

c) For every real number x, $\sqrt{x^2} = x$.

2. (10pts) Negate the following statements.

a) The pillow is blue and fluffy.

b) If $x > 4$, then $x^2 + 2x - 4 < 0$.

c) There exists an $x \in \mathbf{R}$ such that $x^3 + x^2 + x < 0$.

d) For every $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ such that $x < n$.

3. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \vee Q) \implies R) \equiv (P \implies R) \wedge (Q \implies R)$. Do not use a truth table.

4. (12pts) Let $\mathbf{Z}_{10} = \{0, 1, 2, \ldots, 8, 9\}$ be the universal set. Let $A = \{0, 1, 4, 9\}$, $B = \{x \in \mathbf{Z}_{10} \mid x \equiv 0 (\text{mod } 3\}$, $C = \{x \in \mathbf{Z}_{10} \mid x \equiv 1 \ (\text{mod } 2)\}$. Use the roster method to write the following sets:

$A \cap C$ $\qquad$ $A \cup C$ $\qquad$ $A - B$ $\qquad$ $B - C$ $\qquad$ B^c $\qquad$ $A \cap B \cap C$

5. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets.
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$A - (B \cap C)$ $\qquad$ $(A - C) \cap B$ $\qquad$ $(A \cap B) - C$

6. (14pts) Prove that an integer n is divisible by 5 if and only if $n^2 - 5n$ is divisible by 5.

7. (10pts) Let x be a real number, and p and q rational numbers. Prove: if x is irrational, then $p + \dfrac{q}{x}$ is irrational.

8. (16pts) Let a and b be integers. Prove or disprove the following statements.
a) If ab is divisible by 4, then a is divisible by 4 or b is divisible by 4.
b) If ab is divisible by 3, then a is divisible by 3 or b is divisible by 3.

9. (12pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $f(x) = 4x \pmod 5$.
a) Write the table of function values.
b) What is the set of preimages of 2?
c) Is f injective? Justify.
d) Is f surjective? Justify.

10. (16pts) Let $h(x) = \dfrac{3x-2}{4x+1}$.

a) What subset of real numbers is the natural domain for this function?

b) Show that this function is injective.

c) What is the range of this function? Justify your answer.

d) What should be the domain and codomain if we want h to be a bijection?

11. (14pts) Use induction to prove: for every $n \in \mathbf{N}$, $1+5+9+\cdots+(4n-3) = n(2n-1)$.

12. (10pts) Let $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be given by $g(m, n) = (-2m + 9n, m - 5n)$. Show that g is surjective.

Bonus. (7pts) Let $x, y \geq 0$ be real numbers. Show that $\sqrt{xy} \leq \dfrac{x+y}{2}$.

Bonus. (8pts) Let A, B be subsets of a universal set U. Prove: if $A - B = B - A$, then $A = B$.