Mathematical Reasoning — Final Exam
MAT 312, Fall 2011 — D. Ivanšić

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Show all your work!

1. (12pts) Are the following statements true or false? Justify.

a) If an integer is divisible by 6, then it is divisible by 3. True

True, If $n=6q$, then $n=6q=3\cdot 2q$, so n is divisible by 3

b) If $\underbrace{\text{roses are blue}}_{\text{false}}$, then $\underbrace{\text{violets are red}}_{\text{false}}$.

True, because false $\Rightarrow$ false is a true statement.

c) For every real number x, $\sqrt{x^2} = x$.

False. $\sqrt{(-3)^2} = \sqrt{9} = 3$ $\sqrt{x^2} = |x|$ in general

2. (10pts) Negate the following statements.

a) The pillow is blue and fluffy.

The pillow is not blue or the pillow is not fluffy.

b) If $x > 4$, then $x^2 + 2x - 4 < 0$.

$x>4$ and $x^2+2x-4 \geq 0$

c) There exists an $x \in \mathbf{R}$ such that $x^3 + x^2 + x < 0$.

For every $x \in \mathbb{R}$, $x^3+x^2+x \geq 0$

d) For every $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$ such that $x < n$.

There exists an $x \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $x \geq n$

3. (12pts) Use previously proven logical equivalences to prove the equivalence $(P \vee Q) \implies R) \equiv (P \implies R) \wedge (Q \implies R)$. Do not use a truth table.

$(P \vee Q \Rightarrow R) \equiv \neg(P \vee Q) \vee R \equiv (\neg P \wedge \neg Q) \vee R \equiv (\neg P \vee R) \wedge (\neg Q \vee R)$

$\equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$

4. (12pts) Let $\mathbf{Z}_{10} = \{0, 1, 2, \ldots, 8, 9\}$ be the universal set. Let $A = \{0, 1, 4, 9\}$, $B = \{x \in \mathbf{Z}_{10} \mid x \equiv 0 (\text{mod } 3\}$, $C = \{x \in \mathbf{Z}_{10} \mid x \equiv 1 \pmod 2\}$. Use the roster method to write the following sets:

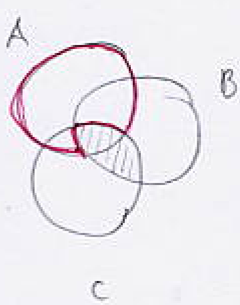
$A \cap C$	$A \cup C$	$A - B$	$B - C$	B^c	$A \cap B \cap C$
$\{1, 9\}$	$\{0, 1, 3, 4, 5, 7, 9\}$	$\{1, 4\}$	$\{0, 6\}$	$\{1, 2, 4, 5, 7, 8\}$	$\{9\}$

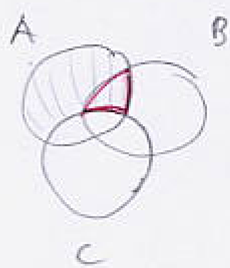
$A = \{0, 1, 4, 9\}$ $B = \{0, 3, 6, 9\}$ $C = \{1, 3, 5, 7, 9\}$

5. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets.
b) Among the three sets, two are equal. Use set algebra to show they are equal.

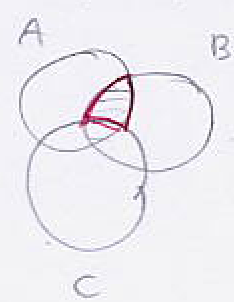
$A - (B \cap C)$ $(A - C) \cap B$ $(A \cap B) - C$

a)





these are $\longleftrightarrow$ equal



b) $(A - C) \cap B = A \cap C^c \cap B = A \cap B \cap C^c = (A \cap B) - C$

$\Longleftrightarrow$

6. (14pts) Prove that an integer n is divisible by 5 if and only if $n^2 - 5n$ is divisible by 5.

To prove a biconditional statement, we prove both implications.

$\Rightarrow$) Suppose n is divisible by 5. Then $n = 5k$ for some integer k.

Then $n^2 - 5n = (5k)^2 - 5 \cdot 5k = 25k^2 - 25k = 5(5k^2 - 5k) = 5k_1$, where

$k_1 = 5k^2 - 5k$. Thus $n^2 - 5n$ has form $5k_1$, so is divisible by 5.

$\Leftarrow$) Consider cases depending on what n is congruent to (mod 5)

$n \equiv r \pmod 5$ where $r = 0, 1, 2, 3, 4$

$n \equiv \square$	$n^2 - 5n$	$\equiv \square \pmod 5$
0	0	0
1	−4	1
2	−6	4
3	−6	4
4	−4	1

From table, we see that if $n^2 - 5n \equiv 0 \pmod 5$

then $n \equiv 0 \pmod 5$,

which means n is divisible by 5.

$q \neq 0$

7. (10pts) Let x be a real number, and p and q rational numbers. Prove: if x is irrational, then $p + \dfrac{q}{x}$ is irrational.

We prove the contrapositive: if $p + \frac{q}{x}$ is rational, then x is rational.

Let $p + \frac{q}{x} = r$, $r \in \mathbb{Q}$. Then

$$\frac{q}{x} = r - p$$

$$\frac{x}{q} = \frac{1}{r-p}$$

(note: if $r = p$, $\frac{q}{x} = 0$, which is not possible)

$x = \dfrac{q}{r-p}$ which is rational, since p, q, r are rational

8. (16pts) Let a and b be integers. Prove or disprove the following statements.
a) If ab is divisible by 4, then a is divisible by 4 or b is divisible by 4.
b) If ab is divisible by 3, then a is divisible by 3 or b is divisible by 3.

a) Let $a=2$, $b=2$. $ab=4$ so ab is divisible by 4, but a is not div. by 4 and b is not divisible by 4.

b) Prove the contrapositive: If a and b are not divisible by 3 then ab is not divisible by 3.

Consider cases with regard to what a, b are congruent to mod 3

$a\equiv$ \ $b\equiv$	1	2
1	1	2
2	2	1

$ab \equiv \square \bmod 3$

We see that in every case $ab \equiv r \pmod 3$ where $r=1$ or 2, never 0.
Hence ab is not divisible by 3.

9. (12pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$, $f(x) = 4x \pmod 5$.
a) Write the table of function values.
b) What is the set of preimages of 2?
c) Is f injective? Justify.
d) Is f surjective? Justify.

a)

x	$4x$	$4x \pmod 5$
0	0	0
1	4	4
2	8	3
3	12	2
4	16	1

b) preimages of $2 = \{3\}$

c) We see from table that when $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$

d) range$(f) = \{0, 1, 2, 3, 4\}$ = codomain(f), so f is surjective.

10. (16pts) Let $h(x) = \dfrac{3x-2}{4x+1}$.

a) What subset of real numbers is the natural domain for this function?
b) Show that this function is injective.
c) What is the range of this function? Justify your answer.
d) What should be the domain and codomain if we want h to be a bijection?

a) $4x+1=0$

$x=-\frac{1}{4}$

domain $= \{x \mid x \neq -\frac{1}{4}\}$

b) Suppose $h(x_1)=h(x_2)$

$\dfrac{3x_1-2}{4x_1+1} = \dfrac{3x_2-2}{4x_2+1} \quad \Big| \cdot (4x_1+1)(4x_2+1)$

$(3x_1-2)(4x_2+1) = (3x_2-2)(4x_1+1)$

$12\cancel{x_1}x_2 - 8x_2 + 3x_1 - \cancel{2} = 12\cancel{x_1}x_2 - 8x_1 + 3x_2 - \cancel{2}$

$11x_1 = 11x_2$ so h is injective.

$x_1 = x_2$

c) $y = \dfrac{3x-2}{4x+1}$

$y(4x+1) = 3x-2$

$4xy + 4y = 3x-2$

$4xy - 3x = -4y-2$

$x(4y-3) = -4y-2$

Turns into $x \cdot 0 = -5$ when $y=\frac{3}{4}$, no solution

$x = \dfrac{-4y-2}{4y-3}$ has solution when $y \neq \frac{3}{4}$

Thus: range$(h) = \{y \mid y \neq \frac{3}{4}\}$

d) Make dom $f = \{x \mid x \neq -\frac{1}{4}\}$

codom $f = \{y \mid y \neq \frac{3}{4}\} =$ range(f)

11. (14pts) Use induction to prove: for every $n \in \mathbf{N}$, $1+5+9+\cdots+(4n-3) = n(2n-1)$.

Base step: $n=1$ $\quad 1 \overset{?}{=} 1\cdot(2\cdot1-1)$ is true

Inductive step: Suppose $1+5+9+\dots+4k-3 = k(2k-1)$ $\quad | + 4(k+1)-3$

$1+5+\dots+4k-3+4k+1 = 2k^2-k+4k+1$

$= 2k^2+3k+1$

$= (k+1)(2k+1)$

$1+5+\dots+4k-3+4(k+1)-3 = (k+1)(2(k+1)-1)$

which is $P(k+1)$

12. (10pts) Let $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be given by $g(m, n) = (-2m + 9n, m - 5n)$. Show that g is surjective.

Let $(s,t) \in \mathbb{Z}\times\mathbb{Z}$. Need to solve:

$$-2m+9n=s$$
$$m-5n=t$$

$m=5n+t$ so

$$-2(5n+t)+9n=s$$
$$-10n-2t+9n=s$$
$$-2t-s=n$$
$$m=5(-2t-s)+t=-10t-5s+t=-9t-5s$$

Since for every $s,t\in\mathbb{Z}$

$m=-9t-5s$, $n=-2t-s$ are both integers,

$g(m,n)=(s,t)$ always has a solution, so g is surjective.

Bonus. (7pts) Let $x, y \geq 0$ be real numbers. Show that $\sqrt{xy} \leq \dfrac{x+y}{2}$.

Investigate:

$$\sqrt{xy} \le \frac{x+y}{2}$$
$$2\sqrt{xy} \le x+y \quad |^2$$
$$4xy \le x^2+2xy+y^2$$
$$0 \le x^2-2xy+y^2$$
$$0 \le (x-y)^2$$

Proof: For every x,y

$$(x-y)^2 \geq 0$$
$$x^2-2xy+y^2 \geq 0 \quad |+4xy$$
$$x^2+2xy+y^2 \geq 4xy$$
$$(x+y)^2 \geq 4xy \quad \text{since } xy\geq 0,\ x+y\geq 0 \text{ may take root}$$
$$x+y \geq 2\sqrt{xy} \qquad \frac{x+y}{2} \geq \sqrt{xy}$$

Bonus. (8pts) Let A, B be subsets of a universal set U. Prove: if $A - B = B - A$, then $A = B$.

Suppose $A-B=B-A$. Let $x\in A$. Assume that $x\notin B$. Then $x\in A$ and $x\notin B$, so $x\in A-B$. Since $A-B=B-A$, $x\in B-A$, i.e. $x\in B$ and $x\notin A$, which contradicts $x\in A$. Thus, $x\notin B$ is false, so $x\in B$, which proves $A\subseteq B$. The other direction, $B\subseteq A$, is proved in the same way.

Or, by set algebra: $A=A\cap(B\cup B^c)=(A\cap B)\cup\underbrace{(A\cap B^c)}_{A-B}=(A\cap B)\cup\underbrace{(A^c\cap B)}_{B-A}=(A\cup A^c)\cap B=B$

$A-B \overset{=}{\longleftrightarrow} B-A$